Ig*-CLOSED SETS IN FUZZY IDEAL TOPOLOGICAL SPACES

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Abstract:
In this paper we introduce the notion of Ig*–closed sets, Ig*-open sets in fuzzy ideal topological space and studied some of its basic properties and characterizations.
It shows this class lies between fuzzy closed sets and fuzzy g–closed sets.

Keywords and Phrases *: Ig*–closed sets, Ig*-open.*

1. Introduction

After the introduction of fuzzy sets by Zadeh [18] in 1965 and fuzzy topology by Chang [2] in 1968, several researches were conducted on the generalization of the notions of fuzzy sets and fuzzy topology. The hybridization of fuzzy topology and fuzzy ideal theory was initiated by Mahmoud [6] and Sarkar [12] independently in 1997. They [6, 12] introduced the concept of fuzzy ideal topological spaces as an extension of fuzzy topological spaces and ideal topological spaces.

A nonempty collection of fuzzy sets I of a set X satisfying the conditions:

(i) if $A \in I$ and $B \leq A$, then $B \in I$ (heredity),

(ii) if $A \in I$ and $B \in I$ then $A \cup B \in I$ (finite additivity)

is called a fuzzy ideal on X. The triplex (X, τ, I) denotes a fuzzy ideal topological space with a fuzzy ideal I and fuzzy topology τ [12].

The local function for a fuzzy set A of X with respect to τ and I denoted by $A^*(\tau, I)$ (briefly A^*) in a fuzzy ideal topological space (X, τ, I) is the union of all fuzzy points x_β such that if U is a Q-neighbourhood of x_β and $E \in I$ then for at least one point $y \in X$ for which $U(y) + A(y) - 1 > E(y)$ [12]. The $*$-closure operator of a fuzzy set A denoted by $Cl^*(A)$ in (X, τ, I) defined as $Cl^*(A) = A \cup A^*$. In (X, τ, I) the collection $\tau^*(I)$ is an extension of fuzzy topological space than τ via fuzzy ideal which is constructed by considering the class $\beta = \{U - E : U \in \tau, E \in I\}$ as a base [6,12].

Recently the concepts of fuzzy semi-I-open sets [4], fuzzy α-I-open sets [16], fuzzy γ-I-open sets [3], fuzzy pre-I-open sets [8] and fuzzy δ-I-open sets [17] have been introduced and studied in fuzzy ideal topological spaces. In the present paper we introduce and study the concept of fuzzy I_{g*}-closed sets in fuzzy ideal topological spaces which simultaneously generalizes the concept of I_{g*} -closed sets [11].

2. Preliminaries

Let X be a nonempty set. A family τ of fuzzy sets of X is called a fuzzy topology [2] on X if the null fuzzy set 0 and the whole fuzzy set 1 belongs to τ and τ is closed with respect to any union and finite intersection. If τ is a fuzzy topology on X, then the pair (X, τ) is called a fuzzy topological space. The members of τ are called fuzzy open sets of X and their complements are called fuzzy closed sets. The closure of a fuzzy set A of X denoted by Cl(A), is the intersection of all fuzzy closed sets which contains A. The interior [2] of a fuzzy set A of X denoted by Int(A) is the union of all fuzzy subsets contained in A. A fuzzy set A of a fuzzy topological space (X, τ) is called fuzzy semi-open if there exists a fuzzy open set U in X such that U≤A≤Cl(U) [1]. A fuzzy set A in (X, τ) is said to be quasi-coincident with a fuzzy set B, denoted by AqB, if there exists a point $x \in X$ such that $A(x) + B(x) > 1$ [4]. A fuzzy set V in (X, τ) is called a Q-neighbourhood of a fuzzy point x_β if there exists a fuzzy open set U of X such that $x_\beta qU \leq V$ [4].

Definition 2.1: A fuzzy set A of a fuzzy topological space (X, τ) is called fuzzy generalized closed written as fuzzy g-closed if Cl(A) ≤ O whenever A ≤ O and O is fuzzy open [14].

Definition 2.2: A fuzzy set A of fuzzy ideal topological space (X, τ, I) is said to be fuzzy ∗-closed (resp. fuzzy ∗-dense in itself) if $A^* \leq A$ (resp. $A \leq A^*$) [12].

Definition 2.3: A fuzzy set A of a fuzzy ideal topological space (X, τ, I) is called fuzzy I_g -closed if $A^* \leq U$, whenever $A \leq U$ and U is fuzzy open in X [13].

Lemma 2.1: $A \leq B \Leftrightarrow \rceil(Aq(1-B))$, for every pair of fuzzy sets A and B of X [9].

3. Fuzzy Ig*-closed sets

Definition 3.1: A fuzzy set A of a fuzzy ideal topological space (X, τ, I) is called fuzzy Ig*-closed if $A^* \leq U$, whenever $A \leq U$ and U is fuzzy g-open in X.

Remark 3.1: Every fuzzy ∗-closed set of a fuzzy ideal topological space (X, τ, I) is fuzzy Ig*-closed and every fuzzy Ig*-closed is fuzzy I_g-closed set but the converse may not be true.

Remark 3.2: In a fuzzy ideal topological space (X, τ, I), I is fuzzy Ig*-closed for every $A \in I$.

Theorem 3.1: Let (X, τ, I) be a fuzzy ideal topological space. Then A^* is fuzzy Ig*-closed for every fuzzy set A of X.

Proof: Let A be a fuzzy set of X and U be any fuzzy g-open set of X such that $A^* \leq U$. Since $(A^*)^* \leq A^*$ it follows that $(A^*)^* \leq U$. Hence A^* is fuzzy Ig*-closed.

Theorem 3.2: Let (X, τ, I) be a fuzzy ideal topological space and A be a fuzzy Ig*-closed and fuzzy g-open set in X. Then A is fuzzy ∗-closed.

Proof: Since A is fuzzy g-open and fuzzy Ig*-closed and $A \leq A$. It follows that $A^* \leq A$ because A is fuzzy Ig*-closed. Hence $Cl^*(A) = A \cup A^* \leq A$ and A is fuzzy ∗-closed.

Theorem 3.3: Let (X, τ, I) be a fuzzy ideal topological space and A be a fuzzy set of X. Then the following are equivalent:

(i) A is fuzzy Ig*-closed.

(ii) $Cl^*(A) \leq U$ whenever $A^* \leq U$ and U is fuzzy g-open in X.

(iii) $\rceil(AqF) \Rightarrow \rceil(Cl^*(A)qF)$ for every fuzzy closed set F of X.

(iv) $\rceil(AqF) \Rightarrow \rceil(A^*qF)$ for every fuzzy closed set F of X.

Proof: (i)⟹(ii). Let A be a fuzzy Ig*-closed set in X. Let $A^* \leq U$ where U is fuzzy g-open set in X. Then $A^* \leq U$. Hence $Cl^*(A) = A \cup A^* \leq U$. Which implies that $Cl^*(A) \leq U$.

(ii)⟹(i). Let A be a fuzzy set of X. By hypothesis $Cl^*(A) \leq U$. Which implies that $A^* \leq U$. Hence A is fuzzy Ig*-closed.

(ii) ⟹(iii). Let F be a fuzzy closed set of X and ⌉(AqF). Then 1−F is fuzzy open in X and by Lemma 2.1, A ≤ 1−F. Therefore, $Cl^*(A)$ ≤ 1−F, because A is fuzzy Ig*-closed. Hence by Lemma 2.1, ⌉($Cl^*(A)$qF).

(iii) ⟹(ii). Let U be a fuzzy Ig*-open set of X such that A^* ≤ U. Then by Lemma 2.1, ⌉(Aq(1−U)) and 1−U is fuzzy closed in X. Therefore by hypothesis ⌉(Cl*(A)q(1−U)). Hence, $Cl^*(A)$ ≤ U.

(i)⟹(iv). Let F be a fuzzy g-closed set in X such that ⌉(AqF). Then A ≤ 1−F where 1−F is fuzzy g-open. Therefore by (i) A^* ≤ 1−F. Hence ⌉(A^*qF).

(iv)⟹(i). Let U be a fuzzy closed set in X such that A ≤ U. Then by Lemma 2.1, ⌉(Aq(1−U)) and 1−U is fuzzy closed in X. Therefore by hypothesis ⌉(A^*q(1−U)). Hence A^* ≤ U and A is fuzzy Ig* -closed set in X.

Theorem 3.4: Let (X, τ, I) be a fuzzy ideal topological space and A be a fuzzy Ig* -closed set. Then x q$Cl^*(A)$ ⟹ Cl(x)qA for any fuzzy point x of X.

Proof: Let x q$Cl^*(A)$. If ⌉(Cl(x)qA). Then by Lemma 2.1, A ≤ (1−Cl(x)). And so by Theorem 3.3(ii), $Cl^*(A)$ ≤ (1−Cl(x)) because (1−Cl(x)) is fuzzy g-open set in X. Which implies that $Cl^*(A)$ ≤(1−x). Hence by Theorem 3.3(ii),⌉(x q$Cl^*(A)$), which is a contradiction.

Theorem 3.5: Let (X, τ, I) be a fuzzy ideal topological space and A be fuzzy ∗-dense in itself fuzzy Ig* -closed set of X. Then A is fuzzy g-closed.

Proof: Let U be a fuzzy open set of X such that A ≤ U. Since A is fuzzy Ig*-closed, by Theorem 3.3 (ii), $Cl^*(A)$ ≤ U. Therefore, Cl(A) ≤ U, because A is fuzzy ∗-dense in itself. Hence A is fuzzy g-closed.

Theorem 3.6: Let (X, τ, I) be a fuzzy ideal topological space where I = {0} and A be a fuzzy set of X. Then A is fuzzy Ig*-closed if and only if A is fuzzy g-closed.

Proof: Since I = {0}, A^* = Cl(A) for each subset A of X. Now the result can be easily proved.

Theorem 3.7: Let (X, τ, I) be a fuzzy ideal topological space and A, B are fuzzy Ig*-closed sets of X. Then AUB is fuzzy Ig*-closed.

Proof: Let U be a fuzzy g-open set of X such that AUB ≤ U. Then A ≤ U and B ≤ U .Therefore $A^* \leq$ U and $B^* \leq$ U because A and B are fuzzy Ig*-closed sets of X. Hence $(AUB)^* \leq$ U and AUB is fuzzy Ig*-closed.

Remark 3.3: The intersection of two fuzzy Ig*-closed sets in a fuzzy ideal topological space (X, τ, I) may not be fuzzy Ig*-closed .

Example 3.1: Let X ={a, b} and A, B be two fuzzy sets defined as follows:

A (a) = 0.9 , A (b) = 0.7
B (a) = 0.8 , B (b) = 0.7
U (a) = 0.3 , U (b) = 0.4

Let τ = { 0, U, 1} and I = {0}. Then A and B are fuzzy Ig*-closed sets in (X, τ, I) but A ∩ B is not fuzzy Ig*-closed.

Theorem 3.8: Let (X, τ, I) be a fuzzy ideal topological space and A, B are fuzzy sets of X such that $A \leq B \leq Cl^*(A)$. If A is fuzzy Ig*-closed set in X, then B is fuzzy Ig*-closed.

Proof: Let U be a fuzzy g-open set such that B ≤ U. Since A ≤ B we have A ≤ U. Hence, $Cl^*(A) \leq U$ because A is fuzzy Ig*-closed. Now $B \leq Cl^*(A)$ implies that $Cl^*(B) \leq Cl^*(A) \leq U$. Consequently B is fuzzy Ig*-closed.

Theorem 3.9: Let (X, τ, I) be a fuzzy ideal topological space and A, B are fuzzy sets of X such that $A \leq B \leq A^*$. Then A and B are fuzzy g-closed.

Proof: Obvious.

Theorem 3.10: Let (X, τ, I) be a fuzzy ideal topological space. If A and B are fuzzy subsets of X such that $A \leq B \leq A^*$ and A is fuzzy Ig*-closed. Then $A^* = B^*$ and B is fuzzy $*$-open in itself .

Proof: Obvious.

Theorem 3.11: Let (X, τ, I) be a fuzzy ideal topological space and $\mathcal{F}$ be the family of all fuzzy $*$- closed sets of X. Then $\tau \subset \mathcal{F}$ if and only if every fuzzy set of X is fuzzy Ig*-closed.

Proof: Necessity. Let $\tau \subset \mathcal{F}$ and U be a fuzzy g-open set in X such that $A^* \leq$ U. Now $U \in \tau \Rightarrow U \in \mathcal{F}$. And so $Cl^*(A) \leq Cl^*(U) = U$ and A is fuzzy Ig*-closed set in X.

Sufficiency. Suppose that every fuzzy set of X is fuzzy Ig*-closed. Let U ∈ τ. Since U is fuzzy Ig*-closed and U≤ U, $Cl^*(U) \leq U$. Hence $Cl^*(U) = U$ and $U \in \mathcal{F}$. Therefore $\tau \subset \mathcal{F}$.

Definition 3.2: A fuzzy set A of a fuzzy ideal topological space (X, τ, I) is called fuzzy Ig*-open if its complement 1−A is fuzzy Ig*-closed.

Remark 3.4: Every fuzzy ∗-open set in a fuzzy ideal topological space (X, τ, I) is fuzzy Ig*-open and every fuzzy Ig*-open is fuzzy I_g-open. But the converse may not be true.

Theorem 3.12: Let (X, τ, I) be a fuzzy ideal topological space and A is fuzzy set of X. Then A is fuzzy Ig*-open if and only if $F \leq Int^*(A)$ whenever F is fuzzy g-closed and $F \leq A$.

Proof: Necessity. Let A be fuzzy Ig*-open and F is fuzzy g-closed set such that $F \leq A$. Then 1−A is fuzzy Ig*-closed, $1-A \leq 1-F$ and 1− F is fuzzy g-open in X. Hence $Cl^*(1-A) \leq (1-F)$. Which implies that $F \leq Int^*(A)$.

Sufficiency. Let U be a fuzzy g-open set such that $1-A \leq U$. Then 1−U is fuzzy g-closed set of X such that $1-U \leq A$. And so by hypothesis, $1-U \leq Int^*(A)$.Which implies that $Cl^*(1-A) \leq U$ and 1−A is fuzzy Ig*-closed. Hence A is fuzzy Ig*-open.

Corollary 3.1: Let (X, τ, I) be a fuzzy ideal topological space and A is fuzzy set of X. Then A is fuzzy Ig*-open if and only if $F \leq Int^*(A)$ whenever F is fuzzy closed and $F \leq A$.

Theorem 3.13: Let (X, τ, I) be a fuzzy ideal topological space and A be a fuzzy set of X. If A is fuzzy Ig*-open and $Int^*(A) \leq B \leq A$, then B is fuzzy Ig*-open.

Proof: Let A be fuzzy Ig*-open in X then 1−A is fuzzy Ig*-closed. Hence $Cl^*(1-A) \leq (1-A)$ is fuzzy g-open set. Also $Int^*(A) \leq Int^*(B) \Rightarrow Cl^*(1-B) \leq Cl^*(1-A)$. Hence, B is fuzzy Ig*-open.

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