# Face and Total Face Product Cordial Labeling in Some Graph Families

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## Abstract

For a planar graph $G$, the vertex labeling function is defined as $g: V(G) \rightarrow \{0, 1\}$ and $g(v)$ is called the label of the vertex $v$ of $G$ under $g$, induced edge labeling function $g^*: E(G) \rightarrow \{0, 1\}$ is given as if $e = uv$ then $g^*(uv) = g(u)g(v)$ and induced face labeling function $g^{**}: F(G) \rightarrow \{0, 1\}$ is given as if $v_1, v_2, \dots, v_n$ and $e_1, e_2, \dots, e_m$ are the vertices and edges of face $f$ then $g^{**}(f) = g(v_1)g(v_2)\dots g(v_n)g^*(e_1)g^*(e_2)\dots g^*(e_m)$. Let us denote $v_g(i)$ is the number of vertices of $G$ having label i under $g$, $e_g(i)$ is the number of edges of G having label i under $g^*$ and $F_g(i)$ is the number of interior faces of $G$ having label i under $g^{**}$, for $i \in \{0, 1\}$. $g$ is called the face product cordial labeling of graph G if $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ $akd$ $|F_g(0) - F_g(1)| \leq 1$. A graph $G$ is a face product cordial graph if it admits face product cordial labeling. Let $g(0)$ and $g(1)$ be the sum of the number of vertices, edges and interior faces having labels 0 and labels 1 respectively. $g$ is called total face product cordial labeling of graph $G$ if $|g(0) - g(1)| \leq 1$. A graph $G$ is called total face product cordial graph if it admits total face product cordial labeling. Face product cordial labeling is investigated for book graph when $k$ is odd and $m \leq 2$. Face product cordial and total face product cordial labeling of gear graph is explored when $k \geq 3$. Also, the graph obtained from a gear graph by duplicating each vertex of degree two by an edge, the graph obtained from a gear graph by duplicating each vertex of degree three by an edge and the graph obtained by duplication of all rim vertices of a gear graph by an edge when $k \leq 3$ is face product cordial and total face product cordial graph.

**Keywords:** Book Graph, Duplication, Face product cordial labeling, Gear graph, Product cordial labeling, Total face product cordial labeling.

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## 1. Introduction

A graph labeling is the assignment of integers to the vertices or edges or both subject to conditions. For different graph labeling techniques we use a dynamic survey of graph labeling by Gallian [1]. We start with a simple, finite, undirected graph $G = (V(G), E(G))$ with vertex set $V$ and edge set $E$ of $G$. For different notation and terminology we follow Gross and Yellen [2]. Face Cordial labeling and Total face product cordial labeling were given by P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan [3]. Now we provide brief summary of definitions and other information which are necessary for the present investigations.

## 2. Definitions

**Definition 2.1** A cycle in a graph is a non-empty trail in which only the first and last vertices are equal.

**Definition 2.2** The graph $W_n = C_n + K_1$ is called a wheel graph. The vertex corresponding to $K_1$ is called the apex vertex and the vertices corresponding to Cn are called the rim vertices [1].

**Definition 2.3** A book graph $B(m, n)$ is a graph obtained by identifying n edges taking one edge from each of the n distinct copies of $C_m$, where n is called number of pages of the book $B(m, n)$. The edge obtained by identifying n edges is called the spine or base of the graph $B(m, n)$ [4].

**Definition 2.4** A gear graph $G_n$ is obtained from the wheel graph $W_n$ by adding a vertex between every pair of adjacent vertices in the cycle $C_n$ [5].

**Definition 2.5** A mapping $f : V(G) \to \{0, 1\}$ is called a binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. We denote $v_f(0)$ as the number of vertices with label 0 and $v_f(1)$ as the number of vertices with label 1 [5].

**Definition 2.6** A product cordial labeling of graph G with vertex set V is a function $f : V(G) \to \{0, 1\}$ such that each edge uv is assigned the label f(u)f(v), the number of vertices with label 0 and the number of vertices with label 1 differ by atmost 1 and the number of edge with label 0 and the number of edge with label 1 differ by atmost 1. A graph which admits product cordial labeling is called a product cordial graph. Sundaram, Ponraj and Somasundaram [6] introduced a product cordial labeling.

**Definition 2.7** The neighbourhood of a vertex v of a graph is the set of all vertices adjacent to v. It is denoted by N(v) [5].

**Definition 2.8** Duplication of a vertex of the graph G is the graph G′ obtained from G by adding a new vertex v′ to such that $N(v') = N(v)$ [5].

**Definition 2.9** Duplication of a vertex $v_k$by a new edge $e = v_k, v'_k$ in a graph G produces a new graph G′ such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$. The symbols of duplication of a vertex by a new edge and duplication of an edge by a new vertex were found by Vaidya and Bansara [7].

**Definition 2.10** For a planar graph G, the binary vertex labeling function is defined as $g : V(G) \to \{0, 1\}$ and g(v) is the label of the vertex v of G under g, induced edge labeling function $g^*: E(G) \to \{0, 1\}$ is given as if $e = uv$ then $g^*(uv) = g(u)g(v)$ and induced face labeling function $g^{**} : F(G) \to \{0, 1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of face f then $g^{**}(f) = g(v_1)g(v_2)\ldots g(v_n)g^*(e_1)g^*(e_2)\ldots g^*(e_m)$. Let us denote $v_g(i)$ is the number of vertices of G having label i under g, $e_g(i)$ is the number of edges of G having label i under $g^*$ and $f_g(i)$ is the number of interior faces of G having label i under $g^{**}$ for $i = 1, 2$. g is called the face product cordial labeling of graph G if if $|v_g(0) - v_g(1)| \leq 1$, $|e_g(0) - e_g(1)| \leq 1$ and $|F_g(0) - F_g(1)| \leq 1$. A graph G is a face product cordial graph if it admits face product cordial labeling [3]. P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan [3] introduced a face product cordial labeling.

**Definition 2.11** For a planar graph G, the binary vertex labeling function is defined as $g : V(G) \to \{0, 1\}$ and g(v) is the label of the vertex v of G under g, induced edge labeling function $g^*: E(G) \to \{0, 1\}$ is given as if $e = uv$ then $g^*(uv) = g(u)g(v)$ and induced face labeling function $g^{**} : F(G) \to \{0, 1\}$ is given as if $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_m$ are the vertices and edges of faces f then $g^{**}(f) = g(v_1)g(v_2)\ldots g(v_n)g^*(e_1)g^*(e_2)\ldots g^*(e_m)$. Let g(0) and g(1) be the sum of the number of vertices, edges and interior faces having labels 0 and labels 1 respectively. g is called total face product cordial labeling of graph G if $|g(0) - g(1)| \leq 1$. A graph G is called total face product cordial graph if it admits total face product cordial labeling [3]. P. Lawrence Rozario Raj and R. Lawrence Joseph Manoharan [3] introduced a total face product cordial labeling.

**Note:-** Throughout the article we use $i \in [k]$, whenever $1 \leq i \leq k$.

## 3. Results

**Theorem *3.1*** A book graph $B(m, k)$ is face product cordial graph if both $m$ and $k$ are odd and $m \geq 3$.
**Proof:** Let $B(m, k)$ be the book graph with both $m$ and $k$ are odd with $m \geq 2$ is obtained by identifying $k$ edges taking one edge from each of the $k$ distinct copies of $C_m$, where $k$ is called number of the pages of $B(m, k)$. The edge obtained by identifying $k$ edges is called the spine. Thus, $|V(B(m, k))| = (m - 2)k + 2$, $|E(B(m, k))| = (m - 1)k + 1$ $akd$ $|F(B(m, k))| = k$. Name the vertices of $G$ as follows: $v_0$and $v'_0$ to the end vertices of the spine in $B(m, k)$. Let $v_{11}, v_{21}, v_{31}, \ldots, v_{(m-2)n}$ be the consecutive vertices of $1^{st}$ copy of $C_m$, $2^{nd}$ copy of $C_m$, ..., $k^{th}$ copy of $C_m$ other than the end vertices of the spine.
Define $f : V(B(m, k)) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in \{v_0, v'_0\}; \\ 1, & \text{if } x = v_{ij}, \quad i \in [\frac{m-1}{2}], j \in [\frac{k-1}{2}]; \\ 1, & \text{if } x = v_{ij}, i = 1, j \in [\frac{k+1}{2}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(1) = \frac{(m-2)n+3}{2}, v_f(0) = \frac{(m-2)n+1}{2}, e_f(1) = \frac{(m-1)n}{2}, e_f(0) = \frac{(m-2)n+1}{2}, F_f(1) = \frac{n-1}{2}$ $akd$ $F_f(0) = \frac{n+1}{2}$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ $akd$ $|F_f(0) - F_f(1)| \leq 1$. Hence, B(m, n) is a face product cordial graph when m and n both odd with m ≥ 3 as it admits face product cordial labeling.

**Illustration 3.1**: Face product cordial labeling of $B(5, 3)$ is shown in the Figure 1.

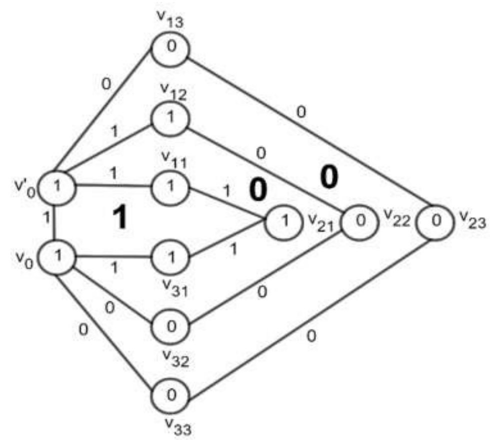


Figure 1: Face product cordial labeling of $B(5, 3)$

**Theorem *3.2*** Gear graph Gn is a face product cordial and total face product cordial graph for n ≥ 3 for odd values of n.

**Proof:** Let $W_n$ be a wheel graph with the apex vertex $v_0$ and consecutive rim vertices as $v_1, v_2, \dots, v_n$. The gear graph $G_n$ is obtained by subdividing each of the rim edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ of $W_n$ by the vertices $u_1, u_2, \dots, u_n$ respectively. Thus, $|V(G_n)| = 2k + 1, |E(G_n)| = 3k$ $akd$ $|F(G_n)| = k$.

Define a function $f : V(G) \to \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = v_0; \\ 1, & \text{if } x = u_i, i \in [\frac{k-1}{2}]; \\ 1, & \text{if } x = v_i, i \in [\frac{k+1}{2}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = k, v_f(1) = k + 1, e_f(0) = \frac{3n+1}{2}, e_f(1) = \frac{3n-1}{2}, F_f(0) = \frac{n+1}{2}, F_f(1) = \frac{n-1}{2}$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ $akd$ $|F_f(0) - F_f(1)| \leq 1$. Hence, $G_n$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = k + \frac{3n+1}{2} + \frac{n+1}{2} = 3k + 1$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (k + 1) + \frac{3n-1}{2} + \frac{n-1}{2} = 3k$. Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

**Illustration 3.2:** Face product cordial labeling and total face product cordial labeling of $G_3$ is shown in the Figure 2.

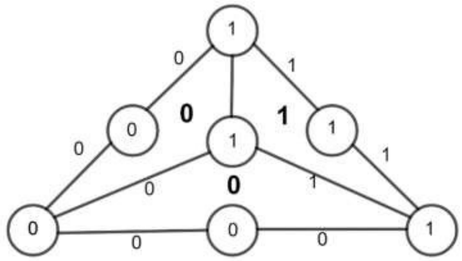

Figure 2: Face prodct cordial labeling of $G_3$ .

**Theorem *3.3*** The graph obtained by duplication of each vertex of degree two by an edge in $G_n$ admits face product cordial labeling and total face product cordial labeling both for $k \geq 3$.
**Proof:** Let $W_n$ be a wheel graph with the apex vertex $v_0$ and consecutive rim vertices as $v_1, v_2, \dots, v_n$. The gear graph $G_n$ is obtained by subdividing each of the rim edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ of $W_n$ by the vertices $u_1, u_2, \dots, u_n$ respectively. Thus, $|V(G_n)| = 2k + 1$, $|E(G_n)| = 3k$ akd $|F(G_n)| = k$. Let $G$ be the graph obtained from $G_n$ by duplicating each vertex $u_i$ of degree two by an edge $u'_iu''_i$ respectively for all $i = 1, 2, 3, \dots, k$. Thus, $|V(G)| = 4k + 1$, $|E(G)| = 6k$ akd $|F(G)| = 2k$.

Define a function $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = v_0; \\ 1, & \text{if } x = v_i, i \in [k]; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 0, & \text{if } x \in \{u'_iu''_i\}, i \in [k]. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 2k, v_f(1) = 2k + 1, e_f(0) = 3k, e_f(1) = 3k, F_f(0) = k, F_f(1) = k$ . Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ akd $|F_f(0) - F_f(1)| \leq 1$. Hence, $G_n$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 2k + 3k + k = 6k$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (2k + 1) + 3k + k = 6k + 1$ . Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

**Illustration 3.3**: Face product cordial labeling and total face product cordial labeling of $G_5$ is shown in the Figure 3.

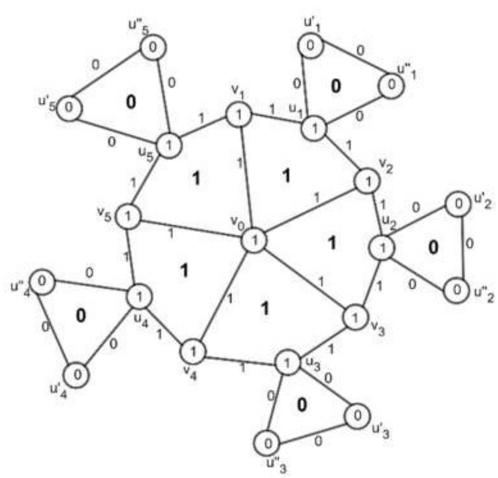

Figure 3: Face prodct cordial labeling of $G_5$ .

**Theorem *3.4*** The graph obtained by duplication of each vertex of degree three by an edge in $G_n$ is both face product cordial graph and total face product cordial graph.

**Proof:** Let $W_n$ be a wheel graph with the apex vertex $v_0$ and consecutive rim vertices as $v_1, v_2, \dots, v_n$. The gear graph $G_n$ is obtained by subdividing each of the rim edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ of $W_n$ by the vertices $u_1, u_2, \dots, u_n$ respectively. Thus, $|V(G_n)| = 2k + 1$, $|E(G_n)| = 3k$ $akd$ $|F(G_n)| = k$. Let $G$ be the graph obtained from $G_n$ by duplicating each vertex $v_i$ of degree three by an edge $v'_iv''_i$ respectively for all $i = 1, 2, 3, \dots, k$. Thus, $|V(G)| = 4k + 1, |E(G)| = 6k$ $akd$ $|F(G)| = 2k$.

Define a function $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = v_0; \\ 1, & \text{if } x = v_i, i \in [k]; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 0, & \text{if } x \in \{v'_iv''_i\}, i \in [k]. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 2k, v_f(1) = 2k + 1, e_f(0) = 3k, e_f(1) = 3k, F_f(0) = k, F_f(1) = k$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ $akd$ $|F_f(0) - F_f(1)| \leq 1$. Hence, $G_n$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 2k + 3k + k = 6k$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (2k + 1) + 3k + k = 6k + 1$. Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

**Illustration 3.4:** Face product cordial labeling and total face product cordial labeling of $G_4$ is shown in the Figure 4.

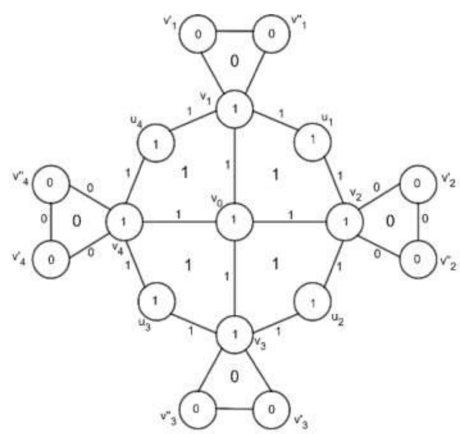

Figure 4: Face prodct cordial labeling of $G_4$

**Theorem** ***3.5*** The graph obtain by duplication of each of the rim vertices in $G_n$ is both face product cordial graph and total face product cordial graph if $k$ is even.

**Proof:** Let $W_n$ be a wheel graph with the apex vertex $v_0$ and consecutive rim vertices as $v_1, v_2, \dots, v_n$. The gear graph $G_n$ is obtained by subdividing each of the rim edges $v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1$ of $W_n$ by the vertices $u_1, u_2, \dots, u_n$ respectively. Thus, $|V(G_n)| = 2k + 1$, $|E(G_n)| = 3k$ $akd$ $|F(G_n)| = k$. Let $G$ be the graph obtained from $G_n$ by duplicating each vertex $v_i$ and $u_i$ by an edge $v'_iv''_i$ and $u'_iu''_i$ respectively for all $i = 1, 2, 3, \dots, k$. Thus, $|V(G)| = 6k + 1, |E(G)| = 9k$ $akd$ $|F(G)| = 3k$.

Case 1: $n \equiv 0 (\mathrm{mod}\ 4)$.

Define a function $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1, & \text{if } x = v_i, i \in \{0, [k]\}; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 1, & \text{if } x = v'_i, i \in [\frac{k}{4}]; \\ 1, & \text{if } x = u'_i, i \in [\frac{k}{4}]; \\ 1, & \text{if } x = u''_i, i \in [\frac{k}{4}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 3k, v_f(1) = 3k+1, e_f(0) = \lceil\frac{9n}{2}\rceil, e_f(1) = \lfloor\frac{9n}{2}\rfloor, F_f(0) = \lceil\frac{3n}{2}\rceil, F_f(1) = \lfloor\frac{3n}{2}\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ and $|F_f(0) - F_f(1)| \leq 1$. Hence, $G^2$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 3k + \lceil\frac{9n}{2}\rceil + \lceil\frac{3n}{2}\rceil = 9k$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (3k+1) + \lfloor\frac{9n}{2}\rfloor + \lfloor\frac{3n}{2}\rfloor = 9k+1$. Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

Case 2: n ≡ 1(mod 4).

Define a function $f : V(G) \rightarrow \{0, 1\}$ such that,

$$f(x) = \begin{cases} 1, & \text{if } x = v_i, i \in \{0, [k]\}; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 1, & \text{if } x = v'_i, i \in [\frac{k+3}{4}]; \\ 1, & \text{if } x = v''_i, i \in [\frac{k-1}{4}]; \\ 1, & \text{if } x = u'_i, i \in [\frac{k-1}{4}]; \\ 1, & \text{if } x = u''_i, i \in [\frac{k-1}{4}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 3k, v_f(1) = 3k+1, e_f(0) = \lceil\frac{9n+1}{2}\rceil, e_f(1) = \lfloor\frac{9n-1}{2}\rfloor, F_f(0) = \lceil\frac{3n+1}{2}\rceil, F_f(1) = \lfloor\frac{3n-1}{2}\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ and $|F_f(0) - F_f(1)| \leq 1$. Hence, $G_n^2$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 3k + \lceil\frac{9n+1}{2}\rceil + \lceil\frac{3n+1}{2}\rceil = 9k+1$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (3k+1) + \lfloor\frac{9n-1}{2}\rfloor + \lfloor\frac{3n-1}{2}\rfloor = 9k$. Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

Case 3: n ≡ 2(mod 4).

Define a function $f : V(G) \rightarrow \{0, 1\}$ such that,

$$f(x) = \begin{cases} 1, & \text{if } x = v_i, i \in \{0, [k]\}; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 1, & \text{if } x = v'_i, i \in [\frac{k+2}{4}]; \\ 1, & \text{if } x = v''_i, i \in [\frac{k+2}{4}]; \\ 1, & \text{if } x = u'_i, i \in [\frac{k-2}{4}]; \\ 1, & \text{if } x = u''_i, i \in [\frac{k-2}{4}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 3k, v_f(1) = 3k+1, e_f(0) = \lceil\frac{9n}{2}\rceil, e_f(1) = \lfloor\frac{9n}{2}\rfloor, F_f(0) = \lceil\frac{3n}{2}\rceil, F_f(1) = \lfloor\frac{3n}{2}\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ and $|F_f(0) - F_f(1)| \leq 1$. Hence, $G^2$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 3k + \lceil\frac{9n}{2}\rceil + \lceil\frac{3n}{2}\rceil = 9k$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (3k+1) + \lfloor\frac{9n}{2}\rfloor + \lfloor\frac{3n}{2}\rfloor = 9k+1$. Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

Case 4: n ≡ 3(mod 4).

Define a function $f : V(G) \rightarrow \{0, 1\}$ such that,

$$f(x) = \begin{cases} 1, & \text{if } x = v_i, i \in \{0, [k]\}; \\ 1, & \text{if } x = u_i, i \in [k]; \\ 1, & \text{if } x = v'_i, i \in [\frac{k+1}{4}]; \\ 1, & \text{if } x = v''_i, i \in [\frac{k+1}{4}]; \\ 1, & \text{if } x = u'_i, i \in [\frac{k+1}{4}]; \\ 1, & \text{if } x = u''_i, i \in [\frac{k-3}{4}]; \\ 0, & otherwise. \end{cases}$$

In view of above labeling pattern we have, $v_f(0) = 3k, v_f(1) = 3k+1, e_f(0) = ]\frac{9n+1}{2}[, e_f(1) = ]\frac{9n-1}{2}], F_f(0) = ]\frac{3n+1}{2}[, F_f(1) = ]\frac{3n-1}{2}]$ . Thus, $|e_f(0) - e_f(1)| \leq 1, |v_f(0) - v_f(1)| \leq 1$ and $|F_f(0) - F_f(1)| \leq 1$. Hence, $G_n$ is face product cordial graph for all odd $k$ and $k \geq 3$. Also, $g(0) = v_f(0) + e_f(0) + F_f(0) = 3k + ]\frac{9n+1}{2}[ + ]\frac{3n+1}{2}[ = 9k+1$ and $g(1) = v_f(1) + e_f(1) + F_f(1) = (3k+1) + ]\frac{9n+1}{2}] + ]\frac{3n-1}{2}] = 9k$ . Thus, $|g(0) - g(1)| \leq 1$. Hence, it is total face product cordial graph.

Thus, by all the above cases it satisfies face and total face product cordial labeling if n is even. Hence the graph obtain by duplication of each rim vertices in $G_n$ admits both face and total face product cordial labeling.

**Illustration 3.4:** Face product cordial labeling and total face product cordial labeling of $G_8$ is shown in the Figure 5.

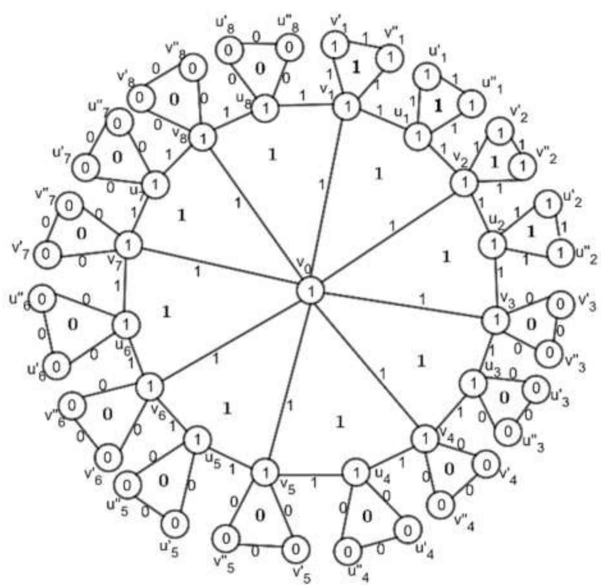

Figure 5: Face product cordial labeling $G_8$

## 4. Conclusion

We applied Face product cordial labeling on Book graph when m and n both odd with $m \geq 3$. Also we applied Face product cordial labeling and Total Face product cordial labeling on Gear graph $(k \geq 3)$. We also showed three results on the graph obtained by switching of a vertex of distinct degrees in the gear graph is Face and Total face cordial labeling under the condition said erlier.

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