STABLE GROUPS

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ABSTRACT. In this paper, we introduce and define the fully stable, minimal(maximal) stable, minimal(maximal) quasi projective, minimal(maximal) duo, fully pseudo stable, minimal pseudo stable, minimal pseudo projective and terse of group G. Next, we obtain some results about them.

1. Introduction

In mathematics, a group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element and that satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility. One of the most familiar examples of a group is the set of integers together with the addition operation, but the abstract formalization of the group axioms, detached as it is from the concrete nature of any particular group and its operation, applies much more widely. It allows entities with highly diverse mathematical origins in abstract algebra and beyond to be handled in a flexible way while retaining their essential structural aspects. The ubiquity of groups in numerous areas within and outside mathematics makes them a central organizing principle of contemporary mathematics. Groups share a fundamental kinship with the notion of symmetry. For example, a symmetry group encodes symmetry features of a geometrical object: the group consists of the set of transformations that leave the object unchanged and the operation of combining two such transformations by performing one after the other. Lie groups are the symmetry groups used in the Standard Model of particle physics; Poincar groups, which are also Lie groups, can express the physical symmetry underlying special relativity; and point groups are used to help understand symmetry phenomena in molecular chemistry. The concept of a group arose from the study of polynomial equations, starting with variste Galois in the 1830s. After contributions from other fields such as number theory and geometry, the group notion was generalized and firmly established around 1870. Modern group theoryan active mathematical discipline studies groups in their own right. To explore groups, mathematicians have devised various notions to break groups into smaller, better-understandable pieces, such as subgroups, quotient groups and simple groups. In addition to their abstract properties, group theorists also study the different ways in which a group can be expressed concretely, both from a point of view of representation theory (that is, through the representations of the group) and of computational group theory. A theory has been developed for finite groups, which culminated with the classification of finite simple groups, completed in 2004. Since the mid-1980s, geometric group theory, which studies finitely generated groups as geometric objects, has become a particularly active area in group theory. In abstract algebra, a normal subgroup is a subgroup which is invariant under conjugation by members of the group of which it is a part. The definition of normal subgroup implies tha

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the sets of left and right cosets coincide. In fact, a seemingly weaker condition that the sets of left and right cosets coincide also implies that the subgroup H of a group G is normal in G. Normal subgroups (and only normal subgroups) can be used to construct quotient groups from a given group. Evariste Galois was the first to realize the importance of the existence of normal subgroups. Epimorphisms are categorical analogues of surjective functions and in the category of sets the concept corresponds to the surjective functions. Many authors in abstract algebra and universal algebra define an epimorphism simply as an onto or surjective homomorphism. Every epimorphism in this algebraic sense is an epimorphism in the sense of category theory, but the converse is not true in all categories. In mathematics, a pseudo group is an extension of the group concept, but one that grew out of the geometric approach of Sophus Lie, rather than out of abstract algebra (such as quasigroup, for example). A theory of pseudogroups was developed by Elie Cartan in the early 1900s. In mathematics, specifically in abstract algebra, a torsion-free abelian group is an abelian group which has no non-trivial torsion elements; that is, a group in which the group operation is commutative and the identity element is the only element with finite order. That is, multiples of any element other than the identity element generate an infinite number of distinct elements of the group. In this work, we introduce the new notions of fully stable, minimal(maximal) stable, minimal(maximal) quasi projective, minimal(maximal) duo, fully pseudo stable, minimal pseudo stable, minimal pseudo projective and terse of group G and investigate some of their properties and structured characteristics.

2. PRELIMINARY

In this section we recall some of the fundamental concepts and definition, which are necessary for this paper. For details we refer reders to [1, 2, 3, 4, 5, 6].

Definition 2.1. A group is a non-empty set G on which there is a binary operation $(a, b) \to ab$ such that
(1) if a and b belong to G then ab is also in G (closure),
(2) $a(bc) = (ab)c$ for all $a, b, c \in G$ (associativity),
(3) there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),
(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$.

Remark 2.2. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \to a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \to ab$ for which the identity is denoted by e.

Example 2.3. (1) $\mathbb{Z}$ with the addition and 0 as identity is an abelian group.
(2) $\mathbb{Z}$ with the multiplication is not a group since there are elements which are not invertible in $\mathbb{Z}$.

Definition 2.4. A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G.

Example 2.5. (1) If we consider the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4, then $H = \{0, 2\}$ is a subgroup of G.
(2) The set of $n \times n$ matrices with real coeffcients and determinant of 1 is a subgroup of $GL_n(R)$, denoted by $SL_n(R)$ and called the special linear group.

Definition 2.6. Given two groups G and H, a group homomorphism is a map $f : G \to H$ such that $f(xy) = f(x)f(y)$ for all $x, y \in G$.

Example 2.7. The map $exp : (\mathbb{R}, +) \to (\mathbb{R}, .)$ with $exp(x) = e^x$ is a group homomorphism.

Definition 2.8. Two groups G and H are isomorphic if there is a group homomorphism $f : G \to H$ which is also a bijection.

Example 2.9. If we consider again the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4 with subgroup $H = \{0, 2\}$ we have that H is isomorphic to $2\mathbb{Z}$, the group of integers modulo 2.

Definition 2.10. Let G be a group and H be subgroup of G. We say that H is a normal subgroup of G, if we have $gH = Hg$ for all $g \in G$.

Example 2.11. Let $GL_n(R)$ be the group of $n \times n$ real invertible matrices, and let $SL_n(R)$ be the subgroup formed by matrices whose determinant is 1. Let $A \in GL_n(R)$ and $B \in SL_n(R)$ then $det(B) = 1$. Now $det(AB) = det(A)det(B) = det(A)1 = 1det(A) = det(B)det(A) = det(BA)$. Therefore $ABA^{-1} \in SL_n(R)$ and so $SL_n(R)$ will be normal subgroup of $GL_n(R)$.

Definition 2.12. Let H be a subgroup of a group G. If $g \in G$, the right coset of H generated by g is $Hg = \{hg, h \in H\}$ and similarly the left coset of H generated by g is $gH = \{gh, h \in H\}$. In additive notation, we get $H + g$ (which usually implies we deal with a commutative group where we do not need to distinguish left and right cosets).

Example 2.13. If we consider the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and its subgroup $H = \{0, 2\}$, the cosets of H in G are $0 + H = H, 1 + H = \{1, 3\}, 2 + H = H, 3 + H = \{1, 3\}$. It is clearly that $0 + H = 2 + H$ and $1 + H = 3 + H$.

Definition 2.14. The group of cosets of a normal subgroup H of G is called the quotient group of G by H and it is denoted by $\dfrac{G}{H}$. The kernel of $f : G \to H$ is defined by $\{g \in G, f(g) = e\}$.

Definition 2.15. Let H be normal subgroup of G. The group homomorphism $\pi : G \to \dfrac{G}{H}$ with $\pi(g) = gH$ is called the natural or canonical map or projection.

Remark 2.16. We have the following terminology: monomorphism=injective homomorphism, epimorphism=surjective homomorphism, isomorphism=bijective homomorphism, endomorphism=homomorphism of a group to itself and automorphism=isomorphism of a group with itself.

Theorem 2.17. *(1st Isomorphism Theorem). If $f : G \to H$ is a homomorphism, then* $\dfrac{G}{\mathrm{Ker}(f)} \cong Im(f)$.

Definition 2.18. An abelian group G is said to be torsion-free if no element other than the identity e is of finite order. Compare this notion to that of a torsion group where every element of the group is of finite order.

3. MAIN RESULTS

Definition 3.1. We say that a group G is fully stable if for each normal subgroup H and for each homomorphism $\alpha : G \to \frac{G}{H}$ we have that $H \subset \mathrm{Ker}(\alpha)$. A normal subgroup H of a group G is stable if for each homomorphism $\alpha : G \to \frac{G}{H}$ we have that $H \subset \mathrm{Ker}(\alpha)$.

Proposition 3.2. *Let K be an arbitrary group and $f, g : G \to K$ are be two homomorphisms such that g is surjective. Then group G is fully stable if and only if* $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$.

Proof. Let group G is fully stable. We prove that $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$. Let $H = \mathrm{Ker}(g)$ and as g is surjective, so we have an isomorphism $\varphi : \dfrac{G}{H} \simeq K$. Now we have $\alpha = \varphi o f : G \to \dfrac{G}{H}$ and so $\mathrm{Ker}(\alpha) = \mathrm{Ker}(f)$. Since G is fully stable so $H \subset \mathrm{Ker}(\alpha) = \mathrm{Ker}(f)$ and so $\mathrm{Ker}(g) \subset \mathrm{Ker}(f)$. Conversely, let H is a subgroup of G and $\alpha : G \to \frac{G}{H}$ a homomorphism, then $\pi_H : G \to \frac{G}{H}$, the natural epimorphism, is surjective, hence by hypothesis $H = \mathrm{Ker}(\pi_H) \subset \mathrm{Ker}(\alpha)$. Thus group G is fully stable. □

Proposition 3.3. *Let G be a fully stable and $\alpha : G \to \frac{G}{H}$ be an epimorphism. Then*
(1) $H = \mathrm{Ker}(\alpha)$.
(2) $H \subset K$ implies that $\alpha(K) = \frac{K}{H}$ where H and K are subgroups of G.

Proof. (1) Let G be a fully stable and $\alpha, \beta, \pi : G \to \frac{G}{H}$ be three homomorphisms such that α is epimorphism(surjective) and π is the natural map. Then $H \subset \mathrm{Ker}(\alpha)$ and Proposition 3.2 give us that $\mathrm{Ker}(\alpha) \subset \mathrm{Ker}(\beta) = \pi^{-1}(\mathrm{Ker}\varphi) = \pi^{-1}(\bar{0}) = H$ such that $\varphi : \frac{G}{H} \cong \frac{G}{\mathrm{Ker}(\alpha)}$. Therefore $H = \mathrm{Ker}(\alpha)$.
(2) As $H \subset K$ so we get that $\theta : \frac{G}{H} \to \frac{G}{K}$ such that $\theta(xH) = xK$ will be an epimorphism. If $\alpha : G \to \frac{G}{H}$ be an epimorphism, then $\theta o \alpha : G \to \frac{G}{K}$ will be epimorphism. Now from part(1) we obtain that $K = \mathrm{Ker}(\theta o \alpha)$ and so $\alpha^{-1}(\mathrm{Ker}(\theta)) = K$ and then $\alpha^{-1}(\frac{K}{H}) = K$ and $\alpha(K) = \frac{K}{H}$. □

Proposition 3.4. *Let $\psi : G \to K$ be group homomorphism.*
If G be fully stable, then $\psi(G)$ will be fully d-stable.
(2) If K be fully stable, then $\psi^{-1}(K)$ will be fully stable.

Proof. (1) Let G be fully stable then for each normal subgroup H and for each homomorphism $\alpha : G \to \frac{G}{H}$ we have that $H \subset \mathrm{Ker}(\alpha)$. Let $\eta : \psi(G) \to \frac{\psi(G)}{\psi(H)}$ be a homomorphism and we prove that $\psi(H) \subset \mathrm{Ker}(\eta)$. As $H \subset \mathrm{Ker}(\alpha)$ so

$$\begin{aligned}\psi(H) &\subset \psi(\mathrm{Ker}(\alpha)) \\ &= \psi\{g \in G \mid \alpha(g) \in H\} \\ &= \{\psi(g) \in \psi(G) \mid \psi(\alpha)(g) \in \psi(H)\} \\ &= \{\psi(g) \in \psi(G) \mid \eta(\psi)(g) \in \psi(H)\} \\ &= \mathrm{Ker}(\eta).\end{aligned}$$

Thus $\psi(H) \subset \mathrm{Ker}(\eta)$.
(2) Let K be fully d-stable then for each normal subgroup J and for each homomorphism $\alpha : K \to \frac{K}{J}$ we have that $J \subset \mathrm{Ker}(\alpha)$. Assume that $\eta : \psi^{-1}(K) \to \frac{\psi^{-1}(K)}{\psi^{-1}(J)}$ be a homomorphism and we prove that $\psi^{-1}(J) \subset \mathrm{Ker}(\eta)$. Since $J \subset \mathrm{Ker}(\alpha)$ so

$$\begin{aligned}\psi^{-1}(J) &\subset \psi^{-1}(\mathrm{Ker}(\alpha)) \\ &= \psi^{-1}\{k \in K \mid \alpha(k) \in J\} \\ &= \{\psi^{-1}(k) \in \psi^{-1}(K) \mid \psi^{-1}(\alpha)(k) \in \psi^{-1}(J)\} \\ &= \{\psi^{-1}(k) \in \psi^{-1}(K) \mid \eta(\psi^{-1})(k) \in \psi^{-1}(J)\} \\ &= \mathrm{Ker}(\eta).\end{aligned}$$

Therefore $\psi^{-1}(J) \subset \mathrm{Ker}(\eta)$. □

Definition 3.5. Let G be a group.
(1) G is said to be minimal(maximal) stable if each minimal(maximal) subgroup of G is stable.
(2) G is said to be minimal(maximal) quasi projective if for each minimal(maximal) normal subgroup H of G and each $\alpha : G \to \frac{G}{H}$, there exists an endomorphism $h : G \to G$ such that $\pi o h = \alpha$ with $\pi : G \to \frac{G}{H}$ is the natural map.
(3) G is said to be minimal(maximal) duo group if for each endomorphism $f : G \to G$ and each minimal(maximal) normal subgroup H we will have that $f(H) \subset H$.

Proposition 3.6. *If H and K are two normal subgroups of a group G such that each of them is not contained in the other and such that $\frac{G}{H} \cong \frac{G}{K}$. Then both of H and K are not stable.*

Proof. Let $\varphi : \frac{G}{H} \to \frac{G}{K}$ be an isomorphism and $\pi : G \to \frac{G}{H}$ be the natural epimorphism, then $\alpha = \varphi o \pi : G \to \frac{G}{K}$ is a homomorphism with $\mathrm{Ker}(\alpha) = \pi^{-1}(\mathrm{Ker}(\varphi)) = H$. As hypothesis $K \nsubseteq H$ so $K \nsubseteq \mathrm{Ker}(\alpha)$ and then K is not stable. Similarly H is not *stable*. ☐

Corollary 3.7. *If H and K are two distinct maximal normal subgroups of G such that $\frac{G}{H} \cong \frac{G}{K}$. Then both of H and K are not stable.*

Proposition 3.8. *Let G be a group. Then G is minimal (maximal) stable if and only if for each arbitrary group K and any two homomorphisms $f, g : G \to K$ with g surjective and $\mathrm{Ker}(g)$ is minimal, $\mathrm{Ker}(g) \subset \mathrm{Ker} f$ (and $\mathrm{Ker}(g)$ is maximal, $\mathrm{Ker}(g) = \mathrm{Ker}(f)$).*

Proof. The proof is similar to proof of Proposition 3.2. ☐

Proposition 3.9. *A group G is minimal quasi projective if and only if for each group K and any two homomorphisms $f, g : G \to K$ with g surjective and $\mathrm{Ker}(g)$ is minimal, there exists an endomorphism $h : G \to G$ such that $g o h = f$.*

Proof. Let group G is minimal quasi projective. As $g : G \to K$ is surjective so we obtain an isomorphism $\varphi : K \to \dfrac{G}{H}$ with $H = \mathrm{Ker}(g)$ and $\varphi o g = \pi$ where $\pi : G \to \frac{G}{H}$ is the natural map. Now $\varphi o f : G \to \dfrac{G}{H}$ is a homomorphism and since H is minimal subgroup so there exists an endomorphism $H : G \to G$ such that $\pi o h = \varphi o f$ so $(\varphi o g) o h = \varphi o f$ and $\varphi o (g o h) = \varphi o f$ then $g o h = f$. This implies G is minimal quasi projective. Conversely, assume that the condition in the proposition holds , let H be a minimal subgroup of G and $\alpha : G \to \frac{G}{H}$ be a homomorphism, set $K = \frac{G}{H}$, then by the hypothesis there exists an endomorphism$h : G \to G$ such that $\pi o h = \alpha$. Thus G is minimal quasi projective. ☐

Proposition 3.10. *Let G be a torsion free group. If G is maximal stable, then it is maximal quasi projective.*

Proof. Let $\alpha : G \to \frac{G}{H}$ be a homomorphism where H is a maximal subgroup of G. If $\alpha = 0$, then $\pi o f = \pi o 0 = 0 = \alpha$.
If $\alpha \neq 0$, then $H \subset \mathrm{Ker}(\alpha)$ and since H is a maximal subgroup of G so $H = \mathrm{Ker}(\alpha)$. We can set $g_0 \in G$ such that $g_0 \notin H = \mathrm{Ker}(\alpha)$ and then $\alpha(g_0) = gH$ with $g \notin H$ and $g = g_0 h$. Then $\alpha(g_0) = gH = g_0 h H = g_0 H$. Thus we can define endomorphism $f : G \to G$ by $f(g) = g$ for all $g \in G$. Therefore $\pi(f(g_0)) = \pi(g_0) = g_0 H = \alpha(g_0)$ and then $\pi o f = \alpha$. Thus G will be maximal quasi projective. ☐

Corollary 3.11. *Let G be a torsion free group and for every endomorphism $f : G \to G$ and every normal subgroup H of G we have that $f(H) \subset H$. Then G maximal stable if and only if it is maximal quasi projective.*

Definition 3.12. Let G be a group.
(1) G is said to be fully pseudo stable if for each group K and any two epimorphisms $f, g : G \to K$ we have that $\text{Ker} f = \text{Ker} g$.
(2) A normal subgroup H of G is said to be pseudo stable if for each epimorphism $\alpha : G \to \frac{G}{H}$ we have that $\text{Ker}(\alpha) = H$.
(3) G is said to be pseudo duo group if for each surjective endomorphism $f : G \to G$ and each subgroup H we have that $f(H) \subset H$.
(4) G is said to be minimal pseudo stable if for each group K and any two epimorphisms $f, g : G \to K$ such that $\text{Ker}(g)$ is minimal, then $\text{Ker}(f) = \text{Ker}(g)$.
(5) G is said to be minimal pseudo projective if each group K and any two epimorphisms $f, g : G \to K$ with $\text{Ker}(g)$ is minimal, there exists a homomorphism $h : G \to G$ such that $f = goh$.

Proposition 3.13. *Let G be a group.*
(1) If G is fully pseudo stable, then it is pseudo duo group.
(2) If G is pseudo projective and duo group, then it is fully pseudo d-stable.
(3) G is fully pseudo stable, if and only if each of its subgroups is pseudo stable.
(4) G is minimal pseudo projective if and only if for each group K and any two epimorphisms $f, g : G \to K$ with $\ker(g)$ is minimal, there exists an endomorphism $h : G \to G$ such that $goh = f$.

Proof. It is clear from definitions. □

Proposition 3.14. *A maximal subgroup of a group is stable if and only if it is pseudo stable.*

Proof. Assume that G is a group and H be a normal maximal subgroup of G. If H be stable, then it is clear that H will be pseudo stable. Convesely, Let H be pseudo stable and $\alpha : G \to \frac{G}{H}$ be a homomorphism. As H is maxinal so $\frac{G}{H}$ is simple. Then
(1) $\alpha : G \to \frac{G}{H}$ is zero and hence $H \subset G = \text{Ker}(\alpha)$.
(2) $\alpha : G \to \frac{G}{H}$ will be an epimorphism and since H is pseudo stable so $H \subset \text{Ker}(\alpha)$.
Thus H will be stable. □

Corollary 3.15. *Every fully pseudo stable group is maximal stable.*

Corollary 3.16. *Let G be a torsion free group. If G is fully pseudo stable, then it is maximal quasi projective.*

Definition 3.17. We say that G is terse, if for each pair of epimorphisms $f : G \to K$ and $g : G \to L$ where K and L are any two isomorphic we must have $\text{Ker}(f) = \text{Ker}(g)$.

Proposition 3.18. *A group G is terse if and only if it is fully pseudo stable.*

Proof. Let G be terse and $f, g : G \to K$ be two epimorphisms, then $\text{Ker}(f) = \text{Ker}(g)$ and then G will be fully pseudo stable. Conversely, let G be fully pseudo stable and $f : G \to K$ and $g : G \to L$ are two epimorphisms with $\text{Ker}(f) \neq \text{Ker}(g)$ and $K \cong L$. Let $h : K \to L$ be an isomorphism so $hof : G \to L$ will be an epimorphism and then $\text{Ker}(hof) = \text{Ker}(f) \neq \text{Ker}(g)$. This is a contradiction (since G is pseudo fully stable). □

Proposition 3.19. *A homomorphic image of a terse (fully pseudo stable) group is terse (fully pseudo stable).*

Proof. Let $h : G \to K$ be an epimorphisim, where G is a terse group, and let $f : K \to L$ and $g : K \to P$ be two epimorphisms with $\text{Ker}(f) \neq \text{Ker}(g)$ and then $\text{Ker}(foh) \neq \text{Ker}(goh)$. Therefore L and P are not isomorphic. Thus K is terse. □

Open problem. A dualization of the basic definitions in this paper give an equivalent statement to the definition of fully d-stable and fully pseudo d-stable and d-derse of group G. Group G is fully d-stable if $f(H) \subset H$ for each normal subgroup H and for each homomorphism $f : H \to G$. A subgroup H of a G is d-stable if $f(H) \subset H$ for each homomorphism $f : H \to G$. A normal subgroup H of group G is called pseudo d-stable if $f(H) \subset H$ for each homomorphism $f : H \to G$. A group in which all subgroups are pseudo d-stable is called fully pseudo d-stable. A group $G$is said to be d-terse if distinct subgroups of G are not isomorphic. Now by the above conceps, one can investigate and obtain new resuls as we obtained in this paper.

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