SOME OPERATOR INEQUALITIES ON SCHUR PRODUCT AND MATRIX WEIGHTED GEOMETRIC MEAN

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Abstract. We complement some properties on operator inequality related to Schur product. Then we discuss the inequalities around the matrix weighted geometric mean, and give

$$\Phi(B\sharp_t A) \leq \Phi(B)\sharp_t\Phi(A),$$

for any positive and unital linear map Φ. Further, if $mI \leq A, B \leq MI$, then

$$\Phi(A\sharp_t B) \geq \frac{4mM}{(M+m)^2}\Phi(A)\sharp_t\Phi(B).$$

1. Introduction

Among all operator inequalities for positive linear maps, a very classical result is Kadison's inequality [8] saying that if Φ is a unital positive (linear) map, then for every Hermitian A

$$\Phi(A)^2 \leq \Phi(A^2).$$

More generally, one may consider $f(\Phi(A)) \leq$ (resp. $\geq$)$\Phi(f(A))$ for operator convex (resp. concave) function f. For example, if $A \geq 0$, we have

$$\Phi(A^p) \leq \Phi(A)^p, 0 \leq p \leq 1; \quad \Phi(A)^p \leq \Phi(A^p), 1 \leq p \leq 2.$$

Along this line, we refer to [2][3][4][6][11].

In this paper, we give some other operator inequalities for a positive linear map. Let diag(A) denote the diagonal matrix whose diagonal entries are those of A.

Theorem 1.1. *Let Φ be a positive and unital linear map, and f be an operator concave function from $[0,\infty)$ into itself. Then, for any positive matrices A and B, it holds that*

(1) $\Phi(f(A)\circ f(B)) \geq \left[\Phi(f(A)^{-1}\circ f(B)^{-1})\right]^{-1}$;

(2) $\Phi(f(A)\circ f(B)) \geq \Phi(\mathrm{diag}(f(B)^{\frac{1}{2}}))\Phi(\mathrm{diag}f(A)^{-1})^{-1}\Phi(\mathrm{diag}(f(B)^{\frac{1}{2}}))$;

(3) $\Phi\left(\frac{f(A)^{-1}+f(B)^{-1}}{2}\right) \geq \left[\Phi\left(f\left(\frac{A+B}{2}\right)\right)\right]^{-1}$.

Next we discuss matrix weighted geometric mean of two $n\times n$ positive definite matrices A and B, defined by

$$A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}, \quad 0 \leq t \leq 1.$$

When $t=\frac{1}{2}$, $B\sharp_{\frac{1}{2}}A$, simply denoted by $B\sharp A$, is the positive definite solution of the Riccati equation $XA^{-1}X = B$ and was well studied (see [2] [12]).

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For a unital positive linear map Φ and two matrices $A, B > 0$. Ando [1] gave the following inequality:

$$\Phi(A\sharp B) \leq \Phi(A)\sharp\Phi(B).$$

Some authors have already attributed the weighted version to Ando as well (see [10]). A more general situation was considered by Fu in [7], but he build the relation between $\Phi^2(A\sharp B)$ and $(\Phi(A)\sharp\Phi(B))^2$. Recently, Dehghani et al. [5] study another generalization for the matrix power mean.

In this paper, we give a development of Ando type inequality as follows.

Theorem 1.2. *Let Φ be a positive and unital linear map. Then*

$$\Phi(B\natural_t A) \leq \Phi(B)\natural_t\Phi(A).$$

Moreover, if A and B are strictly positive matrices with $mI \leq A, B \leq MI$, then

$$\Phi(A\natural_t B) \geq \frac{4mM}{(M+m)^2}\Phi(A)\natural_t\Phi(B).$$

The bound $\frac{4mM}{(M+m)^2}$ may not be sharp. So it might be interesting to pursue a sharper constant C for $\Phi(A\natural_t B) \geq C\Phi(A)\natural_t\Phi(B)$.

2. The proofs of main results

Proof of Theorem 1.1. (1) Since f is an operator concave function from $[0,\infty)$ into itself, then it is operator monotone. Therefore, $f(A) > 0, f(B) > 0$. Recall that, for any two strictly positive P and Q, the block matrix $\begin{pmatrix} P & X \\ X^* & Q \end{pmatrix}$ is positive if and only if $P \geq XQ^{-1}X^*$. Thus it follows that

$$\begin{pmatrix} f(A) & I \\ I & f(A)^{-1} \end{pmatrix} \geq 0, \quad \begin{pmatrix} f(B) & I \\ I & f(B)^{-1} \end{pmatrix} \geq 0.$$

This gives

$$\begin{pmatrix} f(A)\circ f(B) & I\circ I \\ I\circ I & f(A)^{-1}\circ f(B)^{-1} \end{pmatrix} \geq 0. \tag{2.1}$$

It known that every positive map Φ has a restricted 2-positive behaviour in the following sense [2]:

$$\begin{pmatrix} P & H \\ H & Q \end{pmatrix} \geq 0 \Longrightarrow \begin{pmatrix} \Phi(P) & \Phi(H) \\ \Phi(H) & \Phi(Q) \end{pmatrix} \geq 0.$$

Again, by using the argument as before, then the result follows from (2.1).

(2) Under the same conditions, we have

$$\begin{pmatrix} f(A) & I \\ I & f(A)^{-1} \end{pmatrix} \geq 0, \quad \begin{pmatrix} f(B) & f(B)^{\frac{1}{2}} \\ f(B)^{\frac{1}{2}} & I \end{pmatrix} \geq 0,$$

and thus

$$\begin{pmatrix} f(A)\circ f(B) & I\circ f(B)^{\frac{1}{2}} \\ I\circ f(B)^{\frac{1}{2}} & I\circ f(A)^{-1} \end{pmatrix} \geq 0.$$

Here we have used $f(B)^* = f(B^*) = f(B)$. So,

$$\begin{aligned} \Phi(f(A)\circ f(B)) &\geq (I\circ f(B)^{\frac{1}{2}})\Phi(I\circ f(A)^{-1})^{-1}\Phi(I\circ f(B)^{\frac{1}{2}}) \\ &= \Phi(\mathrm{diag}(f(B)^{\frac{1}{2}}))\Phi(\mathrm{diag}(f(A)^{-1})^{-1}\Phi(\mathrm{diag}(f(B)^{\frac{1}{2}})). \end{aligned}$$

(3) Since f is operator concave and $A, B > 0$,

$$f\left(\frac{A+B}{2}\right) \geq \frac{f(A)+f(B)}{2}.$$

Noticed that the map $X \to X^{-1}$ is order-reversing and convex on positive operator. This means that

$$\left[f\left(\frac{A+B}{2}\right)\right]^{-1} \leq \left[\frac{f(A)+f(B)}{2}\right]^{-1} \leq \frac{f(A)^{-1}+f(B)^{-1}}{2}.$$

From the above inequality and $\Phi(X^{-1}) \geq \Phi(X)^{-1}$ for $X > 0$, the conclusion holds, as desired. □

Proof of Theorem 1.2. We follow the arguments in [1]. For a positive definite matrix B, set

$$\Psi(X) = \Phi(B)^{-\frac{1}{2}}\Phi(B^{\frac{1}{2}}XB^{\frac{1}{2}})\Phi(B)^{-\frac{1}{2}}. \tag{2.2}$$

Then Ψ is also positive and unital. By Löwner-Heinz theorem and Kadison inequality, we have

$$\Psi(X^t) \leq \Psi(X)^t, \quad 0 \leq t \leq 1.$$

Note that $B\sharp_t A = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t B^{\frac{1}{2}}$. By taking $X = (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t$ in (2.2), we have

$$\begin{aligned}\Phi(B\sharp_t A) &= \Phi(B)^{\frac{1}{2}}\Psi[(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t]\Phi(B)^{\frac{1}{2}}\\ &\leq \Phi(B)^{\frac{1}{2}}[\Psi(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})]^t\Phi(B)^{\frac{1}{2}}\\ &= \Phi(B)^{\frac{1}{2}}[\Phi(B)^{-\frac{1}{2}}\Phi(A)\Phi(B)^{-\frac{1}{2}}]^t\Phi(B)^{\frac{1}{2}}\\ &= \Phi(B)\sharp_t\Phi(A).\end{aligned}$$

Next, we are to prove the second inequality. Observe that $(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}$, and $\Phi(X^{-1}) \geq [\Phi(X)]^{-1}$ for any positive definite matrix X. Compute

$$\begin{aligned}\Phi(A\sharp_t B) &= \Phi[(A^{-1}\sharp_t B^{-1})^{-1}]\\ &\geq [\Phi(A^{-1}\sharp_t B^{-1})]^{-1}\\ &\geq [\Phi(A^{-1})\sharp_t\Phi(B^{-1})]^{-1}\\ &= [\Phi(A^{-1})]^{-1}\sharp_t\Phi[(B^{-1})]^{-1}. \end{aligned}\tag{2.3}$$

Recall that if A satisfies $mI \leq A \leq MI$, we have (see [2], p.56)

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4mM}\Phi(A)^{-1}.$$

Then

$$\begin{aligned}[\Phi(A^{-1})]^{-1}\sharp_t\Phi[(B^{-1})]^{-1} &\geq \left(\frac{4mM}{(M+m)^2}\Phi(A)\right)\sharp_t\left(\frac{4mM}{(M+m)^2}\Phi(B)\right)\\ &= \frac{4mM}{(M+m)^2}\Phi(A)\sharp_t\Phi(B).\end{aligned}$$

This together with (2.3) yards

$$\Phi(A\sharp_t B) \geq \frac{4mM}{(M+m)^2}\Phi(A)\sharp_t\Phi(B).$$

□

In ([9], Lemma 2.1), we have

$$A\sharp_t B \leq (1-t)A + tB, \quad 0 \leq t \leq 1,$$

for positive definite matrices A and B. It is called the noncommutative weighted arithmetic meangeometric mean inequality.

Corollary 2.1. *Let Φ be positive and unital. If A and B are strictly positive matrices with $mI \leq A, B \leq MI$, then*

$$\Phi((1-t)A + tB) \geq \frac{4mM}{(M+m)^2}\Phi(A)\sharp_t\Phi(B).$$

Competing interests

The author declare that they have no competing interests.

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