An Insight into Incoming University Students' Understanding of Mathematical Syntax: A Case Study

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ABSTRACT This paper provides a quantitative insight on a sample (n = 232) of incoming university students' understanding of mathematical syntax. Questions were provided to them in the form of a voluntary online diagnostic quiz. The quiz focused on items that are vital for communicating mathematical ideas. In particular it examined the abilities of the students' to translate information from the verbal to symbolic form and the recognition of synonyms for commonly used mathematical phrases. The former focused on intervals that were bounded while the latter focused on phrases such as: 'arbitrary', 'therefore', 'for all'. It was found that at least 40% of the sample of students lacked understanding of critical mathematical syntax issues expected of incoming students. The study also showed that an online diagnostic facility that allows re-attempts could help students to improve their abilities with regard to the mathematical syntax focused on in this paper. For translations from the verbal to symbolic forms the improvements ranged from 17 to 26%, while for the ability to recognise synonyms for commonly used mathematical phrases the improvements ranged from 2 to 18%.

INTRODUCTION

These researchers began in the year 2012 their exploration of the under-preparedness of University entrants, who enrolled to study mathematics. That exploration was guided by their discussions with colleagues who lectured first year university calculus. These resulted in the documentation of lecturer expectations of student learning outcomes and possible sample diagnostic items in the context of pre-calculus mathematics (Maharaj and Wagh 2014). Those focused on the knowledge and abilities that the university lecturers expected students to have acquired during their grades 10 to 12 schooling. That resulted in setting-up of five online diagnostic quizzes for incoming students at the University of KwaZulu-Natal, in South Africa. The taking of those quizzes were voluntary.

This paper focuses on the analysis of the data for the diagnostics quiz on mathematical syntax, vital for the communication of mathematical ideas. In particular the researchers intended to look at the ability of incoming students to translate verbal information to the equivalent symbolic form. The focus of their planning for this study was to determine the competence of students with reference to common mathematical syntax that they were expected to have mastered during their study of school level mathematics. This was bearing in mind that the relevant document for school level mathematics (Department of Basic Education 2012:8) stated that mathematics “is a language that makes use of symbols and notations for describing numerical, geometric and graphical relationships.”

Research question

The main research question was: What level of understanding do incoming university students have of basic mathematical syntax? To answer this question the following sub-questions were formulated: What is the ability level of students to translate information from verbal to symbolic forms? What is the level of student recognition of commonly used mathematical synonym phrases?

LITERATURE REVIEW

A search on relevant literature for this study revealed that a number of researchers (for example, Mitchelmore and White 2004; Brannon 2005; Baldwin 2009; Friedrich and Friederici 2009; Quinnell and Carter 2012; Kahle and Keller 2015; Rini, Hussen, Hidayati and Muttaqien 2021) focused on the importance of mathematical syntax when teaching or learning mathematics. To promote readability this section focuses on: Mathematical syntax and semantics; Symbols and abbreviations in mathematics.

Mathematical syntax and semantics

A study of relevant literature on these concepts (Nelson 2002; Mitchelmore and White 2004; Brannon 2005; Baldwin 2009; Friedrich and Friederici 2009; Quinnell and Carter 2012; Kahle and Keller 2015) revealed that: mathematics is a universal symbolic formal language system that makes use of symbols to represent ideas; the representation of ideas could include expressions, equations, inequalities or relations; these are formed by stringing together symbols according to accepted rules of formation (Nelson 2002); the latter is within the context of a formal system which includes a deductive system consisting of sets of transformation rules or axioms, or both; these allow for making deductions by the transforming of one or more mathematical idea representations; *mathematical syntax* refers to the structure or form of the mathematical representation while *semantics* refers to giving meaning to or interpreting the mathematical structure or form of the mathematical representation. Easdown (2006) argued that these lie at opposite ends of a spectrum. An analogy for comprehension with regard to mathematical syntax and semantics could be the features of a well of water. The surface of the water in the well could be viewed as representing mathematical syntax which include the symbols, rules of formation and deductive system for the formal language of mathematics. Depending on the depth of the well, below is the bed which represents the semantics. These give the notion of depth of understanding; superficial understanding versus deep understanding. In the context of the teaching and learning situation the challenge is to provide strategies that enable students to move to from the superficial to deep understanding. Nelson (2002) noted that some mathematicians feel that a study of syntax is sufficient while others feel that semantics should also be focused on. That researcher in defence of the latter argued that semantics could be a useful source of inspiration and is essential when viewed from the context of pedagogy. This is in the sense that students who do well in calculus generally have an understanding of meaning attached to their calculations. It is the opinion of these researchers that semantics could enrich the depth of understanding of students, if the analogy of the well of water is accepted. To enable incoming students to correct their thinking with regard to different types of intervals these researchers focused on items involving translation from verbal to symbolic forms. The interested reader is referred to Table 2, to view such items.

Symbols and abbreviations in mathematics

The representation of ideas in mathematics makes use of symbols in many different contexts (Maharaj 2008). This is the essence of the language of mathematics which in turn makes it unique and complex, since it codes ideas and thought patterns. An important feature of one's mathematical register is the ability to appropriately use symbols and abbreviations to communicate ideas. It could be argued that it is possible to think mathematically without the use of symbols. However, it is the correct use of mathematical symbols that concisely conveys the written communication (Quinnell and Carter 2012) of mathematical ideas. Research by Rini, Hussen, Hidayati and Muttaqien (2021) revealed that incoming first year students at an institution in

Malaysia were lacking in their ability to associate symbols to unpack given problems and that the participating students in their understanding problems used words instead of conventional mathematical symbols.

Table 1 summarises some of the symbols and abbreviations used in mathematics. An implication from this is that anyone studying mathematics should have a clear understanding of what is meant by such mathematical symbols and abbreviations. These could be viewed as pre-requisites to comprehend the ideas conveyed and also to communicate ideas to others. The following summary include the ideas noted by Quinnell and Carter (2012): (1) symbols for the 10 numerals and 26 letters of the alphabet should be familiar to even the youngest of students; (2) students are introduced to symbols such as those used for equality (=), currency (R or $) and the basic arithmetic operations at an early stage of their schooling; (3) students encounter the letters of the Greek alphabet (for example π) and symbols that represent more complex mathematical ideas (%, $\sqrt{x^2}$, <, >, ±, ∞) in their later schooling years. Quinnell and Carter (2012) argued that with regard to symbols used in mathematics their recall or recognition is not complex. The challenge to one studying mathematics is to comprehend that the same symbol could also represent different ideas depending in the context in which it is used. For example if one looks at the classification under pronumerals in Table 1 it should be apparent that the positioning of a letter of the alphabet within a mathematical structure comprising of numerals, operators and letters determines the idea that it represents. In the illustrated examples they could represent a constant, unknown, variable or parameter. Students also need to know that as a convention certain letters are normally used to represent each of these concepts. These imply that it is the semantics or meanings assigned to such coding symbols or concepts that could pose difficulties to students. Also, the syntax or the way in which the symbols are used within a mathematical structure could introduce further complexities for a student engaged with the study of mathematics.

Table 1: Some symbols and abbreviations used in mathematics

Classification of symbols	*Examples*
Numerals – used in various combinations to represent numbers	Hindu-Arabic: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 Roman: I, V, X, D, C, L, M
Operators • In arithmetic	 +, −, ×, ÷

• 'synonyms' such as • others	· (dot) for × √ or ! (factorial)
Comparatives • used to denote relationships • their negation forms	 =, <, >, ≤, ≥, ≡, ≈, ∝, ⊂ ≠, ⊄, ≮, ≉
Grouping symbols • which code structure details	 parentheses ( ), braces { }, brackets [ ]
Pronumerals include letters of Greek alphabet • constants • unknown • variable quantities • parameters	 k in $4x + k$ or $k \sin x$ x in $4x + 3 = 0$ or $4 \sin x + 2 = 0$ x, y in $y = 3x - 2$ m, b in $y = mx + b$
Geometry symbols • vertices in triangle ABC • sides in triangle ABC • angles in triangle ABC	 A, B and C AB or *c* $A\hat{B}C$
Shortened forms • mathematical symbols • abbreviations for units of measurement • common use abbreviations	 %, ∴, ∞, *f*( ), ±, ∫, ∃, ∀ cm, m, m^2 N, S, E, W or am, pm

Expanding on the ideas summarised in Table 1 the following should be noted. Different symbols could have the same meaning. For example ten divided by two could be represented by $10 \div 2$ or $\frac{10}{2}$; these are not the only representations. Some symbols are also very similar to other symbols. Subtle differences could include upper or lower case letters, the size or shape of otherwise identical symbols, the use of italics or bold font or the inclusion of marks such as dots or dashes. Additional issues could be the different conventions in countries or even institutions at different levels. For example in South African schools the decimal sign is represented by a comma (,) while some universities use the dot (.) for this representation. One of the reasons for this is that at school level the learning materials in the form of workbooks and textbook are written by locals while the imported textbooks used at some universities are written by overseas authors. Further, subtle differences in mathematical syntax could represent different ideas. For example the idea represented by $\left[(2+4) \times \sqrt{9}\right]^2$ is different from that represented by $\left[2 + 4 \times \sqrt{9}\right]^2$. Also $a \times b$

could be represented by represented by $a.b$ or ab, while cm does not necessarily represent the product of two quantities c and m but the abbreviation for centimetre. Students should be aware of and recognise all of these issues, so that they could use them or decode information correctly.

CONCEPTUAL FRAMEWORK

The conceptual framework was guided by the principles that emerged from the review of literature and these researchers' experience of teaching calculus to first year university students. The formulation of these principles were also informed by a study of literature on outcomes and the purpose of assessment (American Association for Higher Education 1991; Banta 2002; Council of Regional Accrediting Commissions 2004; Maharaj and Wagh 2014). These guiding principles are as follows:

1. It is necessary to formulate and document expected learning outcomes with reference to mathematical syntax, on which the teaching learning activities should focus. These outcomes should be known to the lecturers, tutors as well as the students before the commencement of the course.
2. The identified learning outcomes should inform the development of the tasks that help in development of necessary skills for mathematical syntax.
3. The taking of the online diagnostic quiz on mathematical syntax should be voluntary.
4. The online quiz could serve as a developmental form of assessment for students.
5. The interested student who takes the quiz, based on the feedback provided by the system will identify his or her strengths and weaknesses; if any. For identified weaknesses the student will take the necessary remedial measures to overcome them. These could include focused studying or consulting with a tutor.
6. The learning of mathematics is hierarchical in the context of relevant concepts and abilities. So a student's ability to use mathematical syntax could either promote or hinder the learning of mathematics. An online approach which focuses on the translation of information from the verbal to symbolic form and the recognition of commonly used phrases could help students to improve their abilities to communicate or decode information based on mathematical syntax.

METHODOLOGY

The items were available in the form of a voluntary online quiz to all incoming students enrolled for the main stream mathematics module, Introduction to Calculus, at the University of KwaZulu-Natal. Those questions and the rationale behind their design were reported by Maharaj and Wagh (2014). The quiz consisted of 10 multiply choice (MC) questions on translating information from a verbal to symbolic form. Those questions were based on closed and open intervals which the students were expected to have worked on during their schooling years, up to grade 12 level. The mathematical syntax quiz also consisted of 6 questions on the recognition of commonly used mathematical synonym phrases such as ‘there exists’ and ‘for all’. Those online questions were also in a MC format. Although the taking of the quiz was voluntary, 232 students completed all 16 questions in the quiz. The online system provided immediate feedback to a student, once a response to an item was selected and submitted. A student had the option to reflect on the response submitted and the feedback. In the case of an incorrect response the student also had the option to re-attempt the quiz item. In this way a student could determine his or her strengths and weaknesses, if any, and take appropriate remedial measures. The latter included appropriate self-study or consulting with a tutor. The online system compiled the statistics with regard to student attempts for each of the 16 online quiz items. Included in those statistics were the total number of first correct attempts and also the total number of correct attempts, which included re-attempts in the cases were first attempts were incorrect. Both of those totals were used to determine the level of incoming student understanding of basic mathematical syntax.

RESULT AND DISCUSSION

These are reported on under the following subsections: Verbal to symbolic translation; Mathematical synonym phrase(s). In each case the data is first presented, followed by the findings and discussion.

Verbal to Symbolic Translation

Table 2 summarises the correct first attempts and subsequent total correct attempts, including re-attempts for each of the ten items based on the translation of information from the verbal to symbolic form. Observe that for the first question only 43% of the incoming sample of students got the translation correct. The table indicates that this percentage improved to 69% if re-attempts are considered. This implies that there was an improvement of 26% after students took the necessary appropriate remedial measures. Note that this question was based on an open interval which had an upper bound, 4 in this case. It seems that a significant percentage of the students; ranging from 17% to 21%; were able to recognise their shortcomings after attempting the first question. This is evident from the study of the first attempts for questions 2 to 6 which were based on the translation of information from the verbal to symbolic forms for the focus on open intervals that were either bounded above or below. Also observe that for those question there were also slight increases ranging from 2% to 9% if re-attempts are considered.

Table 2: Percentage correct responses for questions on verbal to symbolic translation (n = 232)

No.	***Question***	***Percentage correct first attempt***	***Percentage correct including re-attempts***
1.	x Is not greater than 4, is represented as ____ or ____.	43	69
2.	x Is not less than 4, is represented as ______ or ______.	61	67
3.	y is at most 5, is represented as ______ or ______.	61	69
4.	y is at least 7 is represented as ______ or ______.	61	69
5.	x is more than 9 is represented as ____ or _____	60	69
6.	x is less than 9 is represented as ____ or _____.	64	66
7.	z is between a and b is represented as _______	49	65
8.	z is between a and b, and includes b, is represented as _______	60	65
9.	u is from a to b is represented as ______.	53	63

10.	u is from a towards b but excludes b, is represented as _______.	53	63

Observe that percentage correct first attempt responses for question 7 was 49%. This question relates to an interval which has both a lower and an upper bound; a and b respectively. The same is true for questions 8 to 10. Although there was a marginal percentage improvement for questions 7 to 10 in the region of 5% to 16%, it seems that for this sample of incoming students verbal to symbolic translations of intervals that had both bounds provided difficulties. These difficulties are related to translations based on the words 'between', 'includes', 'excludes', 'from', 'towards' and their equivalent symbolic representations involving syntax. For example, for question 10 the correct translation is $a \leq u < b$ or $u \in [a, b)$ and not for example $a < u < b$ or $u \in (a, b)$.

Mathematical Synonym Phrase(s)

Table 3 summarises the correct percentage responses for questions 11 to 16 based on the recognition of commonly occurring mathematical synonyms. Here the focus was not on the symbolic representations but rather the equivalent verbal forms. The following observations can be made for the sample of students with regard to first attempts based on the recognition of synonyms:

- *Therefore* - only 55% were able to correctly recognise the equivalent synonyms: hence, implies, thus.
- *Because* – only 41% were able to correctly recognise the equivalent synonyms: as, since.
- *if* – only 48% were able to correctly recognise the equivalent synonyms: given, given that.
- *Arbitrary* – only 47% were able to correctly recognise the equivalent synonyms: for any, for random.
- *For all* – 58% were able to correctly recognise the equivalent synonyms: for every, for any, without exception.
- *There exists* – only 45% were able to correctly recognise the equivalent synonyms: for some, there is.

Observe that for the sample of incoming students, for these commonly used mathematical phrase(s) at least 42% were unable to recognise correctly the equivalent synonyms. This is rather alarming for such very commonly used mathematical phrases and could be a significant hindrance to student progress in their studies if not adequately addressed in their early stages at university. The authors suspect that this could be one of the most serious causes behind large scale failure rates for first year mathematics modules. This is since all of the above mathematical phrases are frequently used in the context of first year university calculus. For example, the phrase(s) ‘arbitrary’, ‘for all’ and ‘there exists’ are used in the delta-epsilon definition of the limit; to communicate the idea of this abstract concept. The words ‘if’, ‘therefore and ‘because’ are frequently used in the setting up of mathematical arguments.

Table 3: Percentage correct responses for recognition of mathematical synonym phrase (n = 232)

No.	***Question***	***Percentage correct first attempt***	***Percentage correct including re-attempts***
11.	The commonly used synonyms for ‘therefore’ are ______ or ______ or ________.	55	62
12.	The commonly used synonyms for ‘because’ are ______ or ______.	41	59
13.	The commonly used synonym for ‘if’ is ______.	48	60
14.	The commonly used synonym phrases for ‘arbitrary’ are ______ or _____	47	59
15.	The commonly used synonym phrases for ‘for all’ are _____ or ______	58	60
16.	The commonly used synonym phrases for ‘there exists’ are _____ or _____	45	59

Table 3 indicates that the provision of the opportunity to re-attempt and rectify shortcomings in the recognition of synonyms for commonly used mathematical phrases led to improvements in the range of 2% to 18% of the sample students. This supports the principles indicated in the conceptual framework that the provision of online diagnostics for mathematical syntax could serve as a developmental form of self-assessment for interested students who are prepared to take the necessary remedial measures, provided they are able to identify their areas of weaknesses.

CONCLUSIONS AND RECOMMENDATIONS

In 2014 at least 40% of the sample of incoming students at the University of KwaZulu-Natal were significantly underprepared in their understanding of mathematical syntax. This was in the context of translations from verbal to symbolic form and the recognition of synonyms for commonly use mathematical phrases. The provision of online diagnostics on mathematical syntax of the types focused on in this this study could lead to a significant improvement of student abilities related to mathematical syntax. For the translation of information from verbal to symbolic forms in the context of open intervals that were bounded either above or below the provision of online feedback and re-attempts resulted in improvements in the range of 17 to 26% for the sample of incoming students, which is substantial. With regard to the recognition of synonyms of commonly used mathematical phrases the re-attempts resulted in improvements of 2 to 18%. The least improvement was for the phrase ‘for all’.

The recommendation is that incoming university students should be provided with online diagnostics and a remedial support system to identify and overcome shortcomings in the context of mathematical syntax required for the study of first year university calculus. Interested role players at school and university levels are welcome to use or improve upon the items indicated in Tables 1 and 2 of this paper. The authors would be interested in knowing about such experiences with students from other institutions.

Authors' Contributions

This research was conceptualised jointly by both the authors. The first author worked mainly on the literature review, research question and methodology. The introduction, conceptual framework, findings and discussion, and conclusions and recommendations were worked on together.

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REFERENCES

American Association for Higher Education 1991. Nine principles of good practice for assessing student learning. Sterling, VA: Stylus.

Baldwin JT 2009. Variables: syntax, semantics and situations. Accessed on 4 October 2015 at http://homepages.math.uic.edu/~jbaldwin/pub/var5.pdf

Banta TW 2002. Characteristics of effective outcomes assessment: Foundations and examples. In: T.W. Banta & Associates. Building a scholarship of assessment. San Francisco, CA: Jossey-Bass.

Council of Regional Accrediting Commissions 2004. Regional accreditation and student learning: A guide for institutions and evaluators. Atlanta: Southern Association of Colleges & Schools. Accessed on 4 October 2015 at http://www.sacscoc.org/pdf/handbooks/GuideForInstitutions.PDF
Brannon EM 2005. The independence of language and mathematical reasoning. Accessed on 4 October 2015 at http://www.ncbi.nlm.nih.gov/pmc/articles/PMC552939/

Department of Basic Education 2012. Curriculum and assessment Policy Statement Grades 10-12 Mathematics. Department of Basic Education, Republic of South Africa.

Easdown E 2006.Teaching mathematics: the gulf between semantics (meaning) and syntax (form). Accessed on 4 October 2015 at http://www.maths.usyd.edu.au/u/pubs/publist/preprints/2006/easdown-13.pdf

Friedrich R and Friederici AD 2009. Mathematical Logic in the Human Brain: Syntax. Accessed on 4 October 2015 at http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2685028/

Kahle R and Keller W 2015. Syntax versus Semantics. Accessed on 4 October 2015 at http://arxiv.org/pdf/1507.04678.pdf

Maharaj A. (2008). Some insights from research literature for teaching and learning mathematics. *South African Journal of Education*. Volume 28, pp.401-414.

Maharaj A & Wagh V 2014. An outline of possible pre-course diagnostics for differential calculus. *South African Journal of Science*, 110(7/8), 27-33.

Nelson E 2002. Syntax and Semantics. Accessed on 4 October 2015 at http://www.math.princeton.edu/~nelson/papers.html

Quinnell L and Carter M 2012. Greek or not The use of symbols and abbreviations in mathematics. Accessed on 4 October 2015 at http://files.eric.ed.gov/fulltext/EJ974986.pdf

Rini, A.D.P., Hussen, S., Hidayati, H., & Muttaqien, A. (2021). Symbol Sense of Mathematics Students in Solving Algebra Problems. *Journal of Physics*: Conference Series. Accessed 8 June 2022 at https://iopscience.iop.org/article/10.1088/1742-6596/1764/1/012114/pdf#:~:text=Symbol%20Sense%20or%20sensibility%20symbols,the%20universal%20set%20%5B1%5D.