Integer solutions of the exponential Diophantine equation $x^{x^n}y^{y^m}z^{z^n}t^{t^m} = w^w(\, m, n > 0)$

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Abstract:

The exponential Diophantine equation under consideration is

$x^{x^n}.y^{y^m}.z^{z^n}t^{t^m} = w^w(\, m, n > 0)$. Analyzed to find the non-zero distinct integral solutions and a few numerical illustrations are presented.

Keywords: Exponential Diophantine equation, integer solutions

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Introduction:

The exponential Diophantine equation$a^x + b^y = c^z$, where x, y and z are positive integers, has been studied for its non-zero integer solutions by several authors [1-5]. In this paper, an interesting form of an exponential Diophantine equation is considered and studied for its integer solution.

Method of Analysis:

The exponential Diophantine equation with five variables to be solved for its non-zero distinct integral solution is

$$x^{x^n}.y^{y^m}.z^{z^n}t^{t^m} = w^w \tag{1}$$

where m and n are positive integer and x, y, z are unknowns

Taking

$$\left.\begin{aligned} x &= a^{\frac{1}{n}}w^{\frac{1}{n}} \\ y &= b^{\frac{1}{m}}w^{\frac{1}{m}} \\ z &= c^{\frac{1}{n}}w^{\frac{1}{n}} \\ t &= d^{1/m}w^{1/m} \end{aligned}\right\} \tag{2}$$

Using (1) and (2) and raising to the power 1/w, it gives

$$w = a^{\left(\frac{a/n}{1-\left(\frac{q}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)}\right)} b^{\left(\frac{b/m}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)}\right)} c^{\left(\frac{c/n}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)}\right)} d^{\left(\frac{d/m}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)}\right)} \quad (3)$$

Setting,

$$\left.\begin{aligned} \frac{\frac{a}{n}}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)} &= -n_1 \\ \frac{\frac{b}{m}}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)} &= -n_2 \\ \frac{c/n}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)} &= -n_3 \\ \frac{d/m}{1-\left(\frac{a}{n}+\frac{b}{m}+\frac{c}{n}+\frac{d}{m}\right)} &= -n_4 \end{aligned}\right\} \quad (4)$$

Solving equation (4) for a, b, c we get,

$$\left.\begin{aligned} a &= \frac{nn_1}{n_1+n_2+n_3+n_4-1} \\ b &= \frac{mn_2}{n_1+n_2+n_3+n_4-1} \\ c &= \frac{nn_3}{n_1+n_2+n_3+n_4-1} \\ d &= \frac{mn_4}{n_1+n_2+n_3+n_4-1} \end{aligned}\right\} \quad (5)$$

In equation (5), $n_1, n_{2,} n_3, n_4$ are natural numbers.

Substituting (5) in (2) amd (3), the corresponding solutions of (1) are given by

$$x = \left(\frac{n_1+n_2+n_3+n_4-1}{nn_1}\right)^{\frac{n_1-1}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_2}\right)^{\frac{n_2}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{nn_3}\right)^{\frac{n_3}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_4}\right)^{\frac{n_4}{n}}$$

$$y = \left(\frac{n_1+n_2+n_3+n_4-1}{nn_1}\right)^{\frac{n_1}{m}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_2}\right)^{\frac{n_2-1}{m}} \left(\frac{n_1+n_2+n_3+n_4-1}{nn_3}\right)^{\frac{n_3}{m}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_4}\right)^{\frac{n_4}{m}}$$

$$z = \left(\frac{n_1+n_2+n_3+n_4-1}{nn_3}\right)^{\frac{n_3-1}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_1}\right)^{\frac{n_1}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_2}\right)^{\frac{n_2}{n}} \left(\frac{n_1+n_2+n_3+n_4-1}{mn_4}\right)^{\frac{n_4}{n}}$$

$$t = \left(\frac{n_1+n_2+n_3+n_4-1}{nn_1}\right)^{\frac{n_1}{m}}\left(\frac{n_1+n_2+n_3+n_4-1}{mn_2}\right)^{\frac{n_2}{m}}\left(\frac{n_1+n_2+n_3+n_4-1}{nn_3}\right)^{\frac{n_3}{m}}\left(\frac{n_1+n_2+n_3+n_4-1}{mn_4}\right)^{\frac{n_4-1}{m}}$$

$$w = \left(\frac{n_1+n_2+n_3+n_4-1}{nn_1}\right)^{n_1}\left(\frac{n_1+n_2+n_3+n_4-1}{mn_2}\right)^{n_2}\left(\frac{n_1+n_2+n_3+n_4-1}{nn_3}\right)^{n_3}\left(\frac{n_1+n_2+n_3+n_4-1}{mn_4}\right)^{n_4}$$

……………………………….(6)

Case: 1

Let m=n=1 and $n_1, n_2, n_3, n_4 = 1$

The corresponding non-zero integral solutions to (1) is given by

$$\left.\begin{aligned} x &= n_1 3^{3n_1} \\ y &= n_1 3^{3n_1} \\ z &= n_1 3^{3n_1} \\ t &= 3^{3n_1} \\ w &= n_1 3^{3n_1+1} \end{aligned}\right\} \quad (7)$$

Numerical Examples:

Table:1

n_1	x	y	z	t	w
1	3^3	3^3	3^3	3^3	3^4
2	2.3^6	2.3^6	2.3^6	3^6	2.3^7
3	3^{10}	3^{10}	3^{10}	3^9	3^{11}
4	4.3^{12}	4.3^{12}	4.3^{12}	3^{12}	4.3^{13}

Case 2

Let m=n=3, $n_4 = 1$ $n_1 = n_2 = n_3 = \propto^3$

The corresponding non-zero integral solution to (1) is given by

$$\left.\begin{aligned} x &= \alpha \\ y &= \alpha \\ z &= \alpha \\ t &= 1 \\ w &= \alpha^3 \end{aligned}\right\} \quad (8)$$

Numerical Examples:

Table:2

α	x	y	z	t	w
3	3	3	3	1	27
4	4	4	4	1	64
5	5	5	5	1	125
6	6	6	6	1	216

Case: 3

Let m=n=1, $n_1 = n_2 = \alpha$ and $n_3 = \alpha, n_4 = 1$

The corresponding non-zero integral solution to (1) is given by

$$\left.\begin{array}{l} x = \alpha 2^{6\alpha} \\ y = \alpha\, 2^{6\alpha} \\ z = 2^{6\alpha+1} \\ t = 3.\,2^{2\alpha} \\ w = 2^{8\alpha} \end{array}\right\} \qquad (9)$$

Numerical Examples:

Table:3

α	x	y	z	t	w
1	2^6	2^6	2^7	3.2^2	2^8
2	2^{13}	2^{13}	2^{13}	3.2^4	2^{16}
3	12.2^{16}.	12.2^{16}.	2^{19}.	3.2^6	2^{24}.
4	2^{26}.	2^{26}.	2^{25}.	3.2^8	2^{32}.
5	5.2^{30}	5.2^{30}	2^{31}.	3.2^{10}	2^{40}.

Case:4

Let m=n=1 and $n_1 = n_2 = n_3 = \alpha^2$, $n_4 = 1$

The corresponding non-zero integer solution to (1) is given by

$$\left.\begin{aligned} x &= 3^{(2\alpha)^2-1} \\ y &= \alpha^2 3^{3\alpha^2} \\ z &= \alpha^2 3^{3\alpha^2} \\ t &= 3^{3\alpha^2} \\ w &== \alpha^2 3^{3\alpha^2+1} \end{aligned}\right\} \tag{10}$$

Numerical Examples:

Table:4

α	x	y	z	t	w
1	80	27	27	27	81
2	$3^{16}-1$	4.3^{12}	4.3^{12}	3^{12}	4.3^{13}
3	$3^{16}-1$	9.3^{27}	9.3^{27}	3^{27}	9.3^{28}
4	$3^{64}-1$	16.3^{48}	16.3^{48}	3^{48}	16.3^{49}
5	$3^{100}-1$	25.3^{75}	25.3^{75}	3^{75}	25.3^{76}

Case:5

Let m=n=1 and $n_1 = n_2 = n_3 = 2\alpha + 1,\ n_4 = 1$

The corresponding non-zero integer solution to (1) is given by

$$x = (2\alpha + 1).\,3^{3(2\alpha+1)}$$

$$y = (2\alpha + 1).\,3^{3(2\alpha+1)}$$

$$z = (2\alpha + 1).\,3^{3(2\alpha+1)}$$

$$t =.\,3^{3(2\alpha+1)}$$

$$w = (2\alpha + 1).\,3^{6\alpha+4}$$

Numerical Examples:

Table:5

α	x	y	z	t	w
1	3^{10}	3^{10}	3^{10}	3^{9}	3^{11}
2	5.3^{15}	5.3^{15}	5.3^{15}	3^{15}	5.3^{16}
3	7.3^{21}	7.3^{21}	7.3^{21}	3^{21}	7.3^{22}
4	3^{29}	3^{29}	3^{29}	3^{36}	3^{30}
5	11.3^{33}	11.3^{33}	11.3^{33}	3^{33}	11.3^{34}

Conclusion:

In this paper an attempt has been made to obtain integer solutions to the exponential equations. Since these equations are rich in variety, one may search for integral solutions to other choices of exponential equations with multiple variables.

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