The study of top predator interference on tri species with "food-limited" model under the toxicant environment: A mathematical implication

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Abstract This study develops and analyzes a non-linear mathematical model to explore the impact of top predator interference on the three species of the marine food chain system, incorporating the growth of "food-limited" prey populations under the toxic effect. The mathematical model is expressed as nonlinear ordinary differential equations involving four state variables: prey density, intermediate predator density, top predator density, and environmental toxicant. The model is being analyzed for stability using the Jacobian matrix and Lyapunov function. The analysis provides adequate constraints for both local and global stability. In the model, it is found that populations of all three species decrease due to the presence of toxicants. Further, it is also observed that as the value of the feeding rate rises, the equilibrium level of all three populations decreases. Finally, our analytical findings are validated through numerical simulations.
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1 Introduction

The study of cooperation between species and their surroundings has drawn a lot of interest, especially from theoretical ecologists, since it is essential to the development of ecosystems. The predator-prey dynamics within the marine food chain have been the focus of several studies, with the findings focusing on the global dynamic behavior, stability analysis, permanence features, and periodic solution of the system. Most research assumes that predators do not hinder each other's activities; consequently, competition among predators is considered to occur only when prey availability is significantly diminished. In reality, numerous scenarios exist where predators must interact with one another, particularly during food acquisition, necessitating resource sharing or competition. Hassell [7] was the first to introduce food chain models that included mutual

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interference. Rosenzweig-MacArthur model [23] describes that in the absence of its only food, the predator population declines rapidly. Several researchers have conducted mathematical investigations on how interference impacts the dynamics of the prey-predator population [9, 14, 19, 21, 25]. R.K. Upadhya et al. [26] investigate the influence of top predator interference on the dynamics of a food chain model that involves both an intermediate and a top predator. It is found that the model has various types of attracting sets, including chaotic behavior. Moreover, the system is stabilized when top predator interference is increased, but it becomes unstable when the residual decline in the top predator population is normalized. D. Jana et al. [10] incorporate the effects of gestation delay and top predator interference on a three-species food chain model's dynamics involving intermediate and top predator populations. They noted irregular behavior when the interference is high or when the gestation period is surpassed its critical value. S. Chakraborty et al. [5] studied a predator-prey system with disease in the prey to explore how predators influence disease dynamics and community composition. Predator interference impacts disease outbreaks and population stability, shaped by habitat and interactions.

With the concept of limited food and space, the food-limited population model was developed by altering the logistic growth equation to consider the average growth rate as a non-linear function of population density. Smith [24] has discovered that species require more food for development and maintenance when they are expanding and less food for maintenance alone once they have achieved their saturation. Several researchers [6, 11, 12] have proposed food-limited population models and have found remarkable findings regarding the Hopf bifurcation and stability of positive solutions. Misra and Babu A. [13] have examined the feasible equilibrium points' local and global stability. Additionally, they have conducted a comparative analysis with the corresponding non-toxicant system and found that toxicants clearly affect the species if the resources are constrained.

Many species confront several kinds of stresses, such as toxicants, which have an impact on their resources, development rate, and carrying capacity. Zhang et al. [28] investigated and created a three-level experimental marine food chain to examine how feeding selectivity affects the movement of methylmercury (MeHg) through the food chain. Raveendra Babu et al. [2] have proposed and examined the dynamical characteristics of a three-species food chain system subjected to toxicant induced stress and found the conditions for survival or extinction of the species. Kavita Yadav et al. [27] considered a marine tri-trophic food chain system with environmental toxicants and distributed delay. They showed that distributed delay and environmental toxicants play a significant role in occurring hopf bifurcation. Some studies have examined tri-trophic food chain systems, focusing on the impact of toxicants on the system's survival or extinction. [3, 4, 15, 16, 17, 18, 20].

In this paper, we formulated a mathematical model to study the interference of top predator on three species involving a "food-limited" system under toxicant stress. The local and global stability of the system are examined. We use the Jacobian matrix to analyze the model's behavior. We also use the Lyapunov function and the Routh Hurwitz criteria to assess the global stability and durability of the system.

2 Mathematical Model

We have examined the food-limited boom of the prey population in a tri-trophic marine food chain subjected to toxicant induced stress. The model presumes that environmental toxins have adverse effects on the prey's development rate and carrying capacity. A basic food-limited growth equation [11, 8] is taken into account for the prey population. The model features a marine food chain system consisting of a prey population, an intermediate predator, and a top predator, all of which follow Holling type-II functional responses. [1, 22]

The state variables of the model are U(T), the density of the prey population; V(T), the density of the intermediate predator population; W(T), the density of the top predator population; and P(T), the concentration of toxicant in the environment.
By considering these as state variables, we construct a mathematical model using the following system of nonlinear ordinary differential equations to examine the impact of a toxicant on a three-species food chain system.

$$\begin{aligned} \frac{dU}{dT} &= UR(P)\left(\frac{K-U}{K+r_0CU}\right)-\left(\frac{A_1U}{B_1+U}\right)\left(\frac{V}{1+W}\right) \\ \frac{dV}{dT} &= C_1\left(\frac{A_1U}{B_1+U}\right)\left(\frac{V}{1+W}\right)-\left(\frac{A_2V}{B_2+V}\right)W-D_1V \\ \frac{dW}{dT} &= C_2\left(\frac{A_2V}{B_2+V}\right)W-D_2W \\ \frac{dP}{dT} &= Q-D_3P-FUP \end{aligned} \tag{1}$$

with $U(0)>0, V(0)>0, W(0)>0, P(0)>0$.
In the model, the first term in the prey equation describes the food-limited growth rate function, which is influenced by the presence of environmental toxicants; the function $R(P)$ signifies the specific growth rate of the prey population, which is adversely impacted by P, K is the natural carrying capacity, C denotes the food-limited parameter, r_0 is the intrinsic growth rate.

In the model $A_ix/(B_i+x)$, $(i=1,2;\ x=U$ and $V)$, the interaction between two species is modeled using a Holling type-II functional response. A_i and B_i $(i=1,2)$ are constants that parametrize the functional response, and D_1 and D_2 are the natural death rates of intermediate and top predator, respectively. Q is the rate of introduction of toxicants into the environment, C_1 and C_2 are the conversion coefficients of intermediate and top predator respectively, D_3 is the decay rate of toxicant in the environment, F is the rate of depletion of the toxicant in the environment due to its intake by the prey population. For our analysis, we have assumed that $R(P)=r_0-r_1P$ where r_1 determines the rate of decrease of the growth rate of prey populations due to the presence of toxicants.
The number of parameters in the system (1) can be simplified using the following scaling transformations:

$$u=\frac{U}{B_1},\quad v=\frac{A_1V}{B_1},\quad w=\frac{A_1W}{A_2B_1},\quad p=\frac{FP}{r_0},\quad t=D_1T.$$

The model takes the following form after rescaling:

$$\frac{du}{dt} = uq_1(1-q_2p)\left(\frac{q_3-u}{q_3+u}\right)-\frac{s_1uv}{(1+u)(1+q_5w)} \tag{2}$$

$$\frac{dv}{dt} = \frac{s_2uv}{(1+u)(1+q_5w)}-\frac{s_3vw}{q_6+v}-v \tag{3}$$

$$\frac{dw}{dt} = \frac{a_1vw}{q_6+v}-a_2w \tag{4}$$

$$\frac{dp}{dt} = a_3-a_4p-a_5up \tag{5}$$

with $u(0)>0, v(0)>0, w(0)>0, p(0)>0$.
Here,

$$q_1=\frac{1}{CD_1B_1},\quad q_2=\frac{r_1}{F},\quad q_3=\frac{K}{B_1},\quad q_4=\frac{K}{Cr_0B_1},\quad q_5=\frac{\gamma_2d_1}{a_1K_1},\quad s_1=\frac{1}{D_1},$$
$$s_2=\frac{A_1C_1}{D_1},\quad s_3=\frac{A_2^2}{D_1},\quad q_6=\frac{A_1B_2}{B_1},\quad a_1=\frac{C_2A_2}{D_1},\quad a_2=\frac{D_2}{D_1},\quad a_3=\frac{FQ}{D_1r_0},$$
$$a_4=\frac{D_3}{D_1},\quad a_5=\frac{FB_1}{D_1}.$$

3 Boundedness of the model

Now we will prove the boundedness.
Theorem 1: The set $\Omega = \{(u,v,w,p) \in R_+^4 : 0 \leq u(t) \leq q_3, 0 \leq s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t) \leq \zeta_1, 0 \leq s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t) + p(t) \geq \zeta_2\}$ where $\zeta_1 = \frac{s_2 q_3 (q_1 q_3 + 1)}{\phi_1}$, $\phi_1 = \min\{1,1,a_2\}$, $\zeta_2 = \frac{a_3}{\phi_2}$, $\phi_2 = \min\{1,1,a_2,a_4\}$ is a region where model is bounded .
Proof: From (2) we get,

$$\frac{du}{dt} \leq u q_1 (1 - q_2 p) \left(\frac{q_3 - u}{q_4 + u}\right) \leq u q_1 (q_3 - u)$$

then, by applying the standard comparison theorem, we get as $t \rightarrow \infty$, $u \leq q_3$
Now, consider the following function:

$$\eta_1(t) = s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t)$$

Using equations (2-4), we obtained

$$\frac{d\eta_1}{dt} + \phi_1 \eta_1 \leq s_2 q_3 (q_1 q_3 + 1)$$

where $\phi_1 = \min\{1,1,a_2\}$ and then by applying the standard comparison as $t \rightarrow \infty$, we get $\eta_1(t) \leq \frac{s_2 q_3 (q_1 q_3 + 1)}{\phi_1}$ and hence,

$$s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t) \leq \zeta_1$$

where $\zeta_1 = \frac{s_2 q_3 (q_1 q_3 + 1)}{\phi_1}$.
Now, let us consider the function:

$$\eta_2(t) = s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t) + p(t)$$

By using the equations (2-5), we get

$$\frac{d\eta_2}{dt} + \phi_2 \eta_2 \geq a_3$$

where $\phi_2 = \min\{1,1,a_2,a_4\}$ and by usual comparison theorem, we get as $t \rightarrow \infty$ $\eta_2(t) \geq \zeta_2$, where $\zeta_2 = \frac{a_3}{\phi_2}$ hence,

$$\eta_2(t) = s_2 u(t) + s_1 v(t) + \frac{s_1 s_3}{a_1} w(t) + p(t) \geq \zeta_2$$

hence the solution of the model is bounded.

4 Analysis of Model

4.1 Equilibrium points

The model has the following four positive equilibrium points, namely, $E_{10}(u,0,0,p)$, $\bar{E_{11}}(\bar{u},0,0,\bar{p})$, $\hat{E_{12}}(\hat{u},\hat{v},0,\hat{p})$, $E_{13}^{\star}(u^{\star},v^{\star},w^{\star},p^{\star})$. We establish the existence of E_{10}, $\bar{E_{11}}$, $\hat{E_{12}}$ and $E_{13}^{\star}$ as follows:

1. **Existence of** $E_{10}(u,0,0,p)$ **and** $\bar{E_{11}}(\bar{u},0,0,\bar{p})$
   From (5) we get,

$$p = \frac{a_3}{a_4 + a_5 u} \tag{6}$$

From (2) we get,

$$u = q_3, \qquad p = \frac{1}{q_2} \tag{7}$$

When $u = q_3$ then by using (6) we get,

$$p = \frac{a_3}{a_4 + a_5 q_3}$$

When $p = \frac{1}{q_2}$ then by using (6) we get,

$$u = \frac{a_3 q_2 - a_4}{a_5}$$

$u > 0$ only if $a_3 q_2 > a_4$.
So from here we get two equilibrium points namely,
$E_{10}\left(q_3, 0, 0, \frac{a_3}{a_4+a_5 q_3}\right)$ and $\bar{E_{11}}\left(\frac{a_3 q_2 - a_4}{a_5}, 0, 0, \frac{1}{q_2}\right)$

Remark 1. The equilibrium point $\bar{E_{11}}$ is locally asymptotically stable under the condition: $a_3 q_2 > a_4$ which implies $\frac{Q r_1}{D_3 r_0} > 1$, from this, we observe that when the ratio of the product of the rate of decrease in prey growth due to the presence of a toxicant to the product of the intrinsic growth rate of the prey and the death rate of the intermediate predator exceeds one, only the prey population will survive, and both predator populations will go extinct.

2. **Existence of** $\hat{E_{12}} = (\hat{u}, \hat{v}, 0, \hat{p})$
From (4) we get,

$$\hat{u} = \frac{1}{s_2 - 1}$$

provided $s_2 > 1$.
By using equation (5) we get,

$$\hat{p} = \frac{a_3(s_2 - 1)}{a_4(s_2 - 1) + a_5}$$

From (2) we get

$$\hat{v} = \frac{s_2 q_1 (1 - q_1 \hat{p})(q_3(s_2 - 1) - 1)}{s_1(s_2 - 1)(q_4(s_2 - 1) + 1))}$$

$\hat{v} > 0$ only if $q_3 > \frac{1}{s_2 - 1}$ and $\hat{p} < \frac{1}{q_1}$.

Remark 2. The equilibrium point $\hat{E_{12}}$ is locally asymptotically stable under the conditions (i) $A_1 C_1 > D_1$ and (ii)$P < r_0/D_1 CF$, from (i), it can be concluded that the product of the intermediate predator's predation rate and its conversion coefficient exceeds the natural death rate of the intermediate predator. From (ii) it can be concluded that environmental toxicant is less than ratio of intrinsic growth rate to the product of food limited constant, natural death rate of intermediate predator and depletion rate of environmental toxicant the only prey and intermediate predator population will survive and the top predator may tend to extinction.

3. **Existence of** $E_{13}^{\star} = (u^{\star}, v^{\star}, w^{\star}, p^{\star})$
From (4) we get,

$$v^{\star} = \frac{a_2 q_6}{a_1 - a_2} \tag{8}$$

exists only if $a_1 > a_2$
From (3) we get,

$$u^\star = \frac{(1+q_5 w)\left(1+\frac{s_3 w}{q_6+v^\star}\right)}{s_2-(1+q_5 w)\left(1+\frac{s_3 w}{q_6+v^\star}\right)} = f_1(w) \tag{9}$$

From (2) we get

$$p^\star = \frac{q_1\left(\frac{q_3-f_1(w)}{q_4+f_1(w)}\right) - \frac{s_1 v^\star}{(1+f_1(w))(1+q_5 w)}}{q_1 q_2\left(\frac{q_3-f_1(w)}{q_4+f_1(w)}\right)} = f_2(w) \tag{10}$$

Let

$$F(w) = a_3 - a_4 f_2(w) - a_5 f_2(w) f_1(w) \tag{11}$$

To show the existence of $E_{13}^\star$, it is sufficient to prove that there is only one positive solution to this equation, for this we may observe that

$$\begin{aligned} F(0) &= a_3 - a_4 * f_2(0) - a_5 f_2(0) f_1(0) \\ &= a_3 - (a_4 + a_5 f_1(0)) f_2(0) > 0 \\ F(k_1) &= a_3 - (a_4 + a_5 f_1(k_1) f_2(k_1) < 0 \end{aligned}$$

This ensures that root of $F(w) = 0$ exists for $0 < w < k_1$, say $\bar{w}$. Moreover, this root will be unique provided

$$F'(w) = -(a_4 + a_5 f_1(w)) f_2'(w) - a_5 f_2(w) f_1'(w) < 0$$

Knowing the value of $w^\star$, the values of $u^\star$ and $p^\star$ can be calculated from equations (9) and (10) respectively.

Remark 3. $E_{13}^\star$ is locally asymptotically stable if $a_1 > a_2$ which implies $C_1 A_2 > D_2$. Here, it is observed that if the natural death rate of the top predator is less than product of the conversion coefficient and interaction coefficient, then all the population will survive.

5 Local stability of the model

The local stability of the equilibrium in the *Model* is determined by calculating the eigenvalues of the Jacobian matrix at each equilibrium point.
The Jacobian matrix associated with the *Model* is

$$J(u,v,w,p) = \begin{bmatrix} a_{11} & -a_{12} & -a_{13} & a_{14} \\ a_{21} & a_{22} & -a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ -a_{41} & 0 & 0 & -a_{44} \end{bmatrix}$$

where,

$$a_{11}=q_1(1-q_1p)\left(\frac{-u^2-2q_4u+q_3q_4}{(q_4+u)^2}\right)-\frac{s_1v}{(1+u)^2(1+q_5w)},\quad a_{12}=\frac{s_1u}{(1+q_5w)(1+u)},$$
$$a_{22}=\frac{s_2u}{(1+u)(1+q_5w)}-\frac{s_2q_6w}{(q_6+v)^2}-1,\qquad a_{23}=\frac{s_2q_5uv}{(1+u)(1+q_5w)^2}+\frac{s_3v}{q_6+v},$$
$$a_{13}=\frac{s_1q_5uv}{(1+u)(1+q_5)^2},\quad a_{14}=q_1^2u\left(\frac{q_3-u}{q_4+u}\right),\quad a_{21}=\frac{s_2v}{(1+u)^2(1+q_5w)},\quad a_{44}=a_5u$$
$$a_{32}=\frac{q_6a_1w}{(q_6+v)^2},\qquad a_{33}=\frac{a_1v}{q_6+v}-a_2,\qquad a_{41}=a_5p.$$

1. For the equilibrium points E_{10} and $\bar{E_{11}}$ characteristic polynomial $|J-\lambda I|$ would be

$$|J-\lambda I| = \left(q_1(1-q_1p)\left(\frac{-q_3^2-q_4q_3}{(q_4+u)^2}\right)-\lambda\right)\left(\frac{s_2q_3}{1+q_3}-1-\lambda\right) \times(-a_2-\lambda)(-a_5q_3-\lambda)$$

   the eigenvalues of the characteristic equation are $-a_2$, $-a_5q_3$, $\frac{s_2q_3}{1+q_3}-1$, $-q_1(1-q_1p)$ $\left(q_3^2+q_4q_3/(q_4+u)^2\right)$ hence the equilibrium points E_{10} and $\bar{E_{11}}$ is locally stable only if $q_3<\frac{1}{s_2-1}$ and $p<\frac{1}{q_1}$.

2. At $\hat{E_{11}}$ one of the eigenvalue of the characteristic equation is $\frac{a_1v}{q_6+v}-a_2$ and the other three eigenvalues are given by the roots of the following cubic equation,

$$\lambda^3+P_1\lambda^2+P_2\lambda+P_3=0 \tag{12}$$

   where,

$$\begin{aligned}
P_1 &= a_5u+1-\frac{s_2u}{1+u}-q_1(1-q_1p)\left(\frac{-u^2-2q_4u+q_3q_4}{(q_4+u)^2}\right)+\frac{s_1v}{(1+u)^2}\\
P_2 &= \frac{s_1s_2uv}{(1+u)^2}+a_5pq_1^2u\left(\frac{q_3-u}{q_4+u}\right)\left(\frac{s_2u}{1+u}-1\right)-a_5u\left(\frac{s_2u}{(1+u)^2}-1\right)+\\
&\quad \left(q_1(1-q_1p)\left(\frac{-u^2-2q_4u+q_3q_4}{(q_4+u)^2}\right)-\frac{s_1v}{(1+u)^2}\right)\left(\frac{s_2u}{1+u}-1-a_5u\right)\\
P_3 &= a_5pq_1^2u\left(\frac{q_3-u}{q_4+u}\right)\left(\frac{s_2u}{1+u}-1\right)-\frac{s_1s_2u^2v}{(1+u)^2}-a_5u\left(\frac{s_2u}{1+u}-1\right)\\
&\quad \left(q_1(1-q_1p)\left(\frac{-u^2-2q_4u+q_3q_4}{(q_4+u)^2}\right)+\frac{s_1v}{(1+u)^2}\right)
\end{aligned}$$

   Based on the Routh-Hurwitz criteria, $\hat{E_{11}}$ exhibits local asymptotic stability if $(a_1-a_2)\hat{v}<q_6a_2$, $a_1>a_2$, $P_1>0$ and $P_1P_2-P_3>0$.

3. The characteristic equation of $E_{13}^{\star}$ as follows:

$$\lambda^4+Q_1\lambda^3+Q_2\lambda^2+Q_3\lambda+Q_4=0 \tag{13}$$

where

$$\begin{aligned}
Q_1 &= c_{44} - c_{11} - c_{33} - c_{22} \\
Q_2 &= c_{33}c_{22} + c_{32}c_{23} - c_{11}c_{44} + (c_{11} - c_{44})(c_{22} + c_{33}) + c_{12} - c_{14}c_{41} \\
Q_3 &= (c_{33}c_{22} + c_{32}c_{23})(c_{44} - c_{11} - c_{11}c_{44}) + (c_{22} + c_{33})(c_{44}c_{11} - c_{14}c_{41}) \\
&\quad - c_{12}c_{21}(c_{33} - c_{44}) + c_{13}c_{21}c_{32} \\
Q_4 &= c_{21}c_{44}(c_{13}c_{32} - c_{12}c_{33}) + c_{14}c_{41}(c_{32}c_{23} + c_{22}a_{33})
\end{aligned}$$

and

$$\begin{aligned}
&c_{11} = q_1(1 - q_1 p^\star)\left(\frac{-u^2 - 2q_4u^\star + q_3q_4}{(q_4 + u^\star)^2}\right) - \frac{s_1v^\star}{(1+u^\star)^2}, \quad c_{44} = a_5u^\star \\
&c_{12} = \frac{s_1u^\star}{(1+q_5w^\star)(1+u^\star)}, \quad c_{13} = \frac{s_1q_5u^\star v^\star}{(1+u^\star)(1+q_5w^\star)^2}, \quad c_{41} = a_5p^\star \\
&c_{14} = q_1^2u^\star\left(\frac{q_3 - u^\star}{q_4 + u^\star}\right), \quad c_{22} = \frac{s_2u^\star}{(1+u^\star)(1+q_5w^\star)} - \frac{s_2q_6w^\star}{(q_6+v^\star)^2} - 1 \\
&c_{21} = \frac{s_2v^\star}{(1+u^\star)^2(1+q_5w^\star)}, \quad c_{23} = \frac{s_2q_5u^\star v^\star}{(1+u^\star)(1+q_5w^\star)^2} + \frac{s_3v^\star}{q_6+v^\star}, \\
&c_{32} = \frac{q_6a_1w^\star}{(q_6+v^\star)^2}, \quad c_{33} = \frac{a_1v^\star}{q_6+v^\star} - a_2
\end{aligned}$$

Based on Routh-Hurwitz criteria, the equilibrium point $E_{13}^\star$ is locally asymptotically stable if

$$Q_i > 0 (i = 1,2,3,4), Q_1Q_2 > Q_3, Q_1Q_2Q_3 > (Q_3^2 + Q_1^2Q_4)$$

Interpreting the findings in ecological terms from these complex expressions is difficult; however, numerical examples are utilized, and graphical representations are provided to illustrate the system's dynamic behavior in relation to equilibrium.

6 Global Stability

If the following inequality holds in the region Ω

$$H_1(q_1(1 - q_2p^\star)(q_3 + q_4)) > H_3s_1v^\star(1 + w^\star) \tag{14}$$
$$H_1H_2 - s_3H_1q_6w^\star - H_2s_2(1+u^\star)q_5w^\star > H_2s_2(1+u^\star)u^\star \tag{15}$$
$$a_2H_2 > a_1v^\star(q_6 + v^\star) \tag{16}$$
$$M_1 + H_3s_1v^\star(1+w^\star) > H_1q_1(1 - q_2p^\star)(q_3 + q_4) \tag{17}$$
$$H_1H_2 - s_3q_6w^\star H_1 > H_2s_2(1+u^\star)(u^\star + q_5w^\star) + \theta \tag{18}$$

where,

$$
\begin{aligned}
G_1 &= \frac{s_1-(u+q_2w(1+v))}{s_2v(1+q_2w)} > 0,\\
H_1 &= (1+u^\star)(1+u)(1+q_5w^\star)(1+q_5w),\\
M_1 &= \frac{H_1[q_1(q_2q_3(u^\star-q_4)+q_2u^\star(q_4+u^\star))-H_3a_5p^\star]^2}{G_3H_3(a_4+a_5u^\star)},\\
M_2 &= [G_1(H_2s_2q_5u^\star v^\star(1+u^\star)+H_1s_3v^\star(q_6+v^\star))-H_1a_1q_6w^\star)]^2,\\
\theta &= \frac{M_2}{H_1(a_2H_2-a_1(q_6+v^\star)v^\star)},\\
H_2 &= (q_6+v)(q_6+v^\star), \qquad H_3=(q_4+u)(q_4+u^\star)
\end{aligned}
$$

then the positive equilibrium $E_{13}^\star$ is globally asymptotically stable with respect to all solutions initiating the interior of the positive region Ω.

Proof: We consider the following positive definite function about $E_{13}^\star$:

$$V_1 = \left(u-u^\star-u^\star\ln\left(\frac{u}{u^\star}\right)\right)+\frac{G_1}{2}(v-v^\star)^2+\frac{G_2}{2}(w-w^\star)^2+\frac{G_3}{2}(p-p^\star)^2$$

Differentiating V_1 with respect to time t, we get

$$\frac{dV_1}{dt} = \left(\frac{u-u^\star}{u}\right)\frac{du}{dt}+G_1(v-v^\star)\frac{dv}{dt}+G_2(w-w^\star)\frac{dw}{dt}+G_3(p-p^\star)\frac{dp}{dt}$$

Using system of equations (2)-(5), after performing some algebraic manipulations, we obtain

$$
\begin{aligned}
\frac{dV}{dt} \;=\;& -(u-u^\star)^2\left(\frac{(1-q_2p)q_1q_3+q_1q_4(1-q_2p^\star)}{H_3}-\frac{s_1v(1+w)}{H_1}\right)\\
&-(v-v^\star)G_1\left(1-\frac{s_3q_6w}{H_2}-\frac{s_2u^\star(1+u)+s_2q_5w(1+u^\star)}{H_1}\right)\\
&-(w-w^\star)^2G_2\left(a_2-\frac{a_1v^\star(q_6+v)}{H_2}\right)-(p-p^\star)^2G_3(a_4+a_5u^\star)\\
&+(u-u^\star)(v-v^\star)\left(\frac{u-s_1+q_5w(1+u)+G_1s_2v(1+q_5w)}{H_1}\right)\\
&-(v-v^\star)(w-w^\star)\left(G_1\left(\frac{(1+u^\star)s_2q_5uv}{H_1}+\frac{s_3v^\star(q_6+v)}{H_2}\right)-\frac{G_2a_1q_6w}{H_2}\right)\\
&+(u-u^\star)(p-p^\star)\left(\frac{q_1}{H_3}(q_2q_3(u-q_4)+q_2u(q_4+u^\star))-G_3a_5p\right)
\end{aligned}
$$

Now $\frac{dV_1}{dt}$ can further be written as sum of the quadratic forms as

$$
\begin{aligned}
\frac{dV}{dt} \;\leq\;& -[(b_{11}/2)(u-u^\star)^2-b_{14}(u-u^\star)(p-p^\star)+(b_{44}/2)(p-p^\star)^2+(b_{22}/2)(v-v^\star)^2\\
&+b_{23}(v-v^\star)(w-w^\star)+(b_{33}/2)(w-w^\star)^2]
\end{aligned}
$$

where,

$$
\begin{aligned}
b_{11} &= \frac{(q_1(1-q_2p)q_3+q_1q_4(1-q_2p^\star))}{H_3} - \frac{s_1v(1+w)}{H_1},\\
b_{22} &= \frac{1-s_3q_6w}{H_2} - \frac{s_2u^\star(1+u)+s_2q_5w(1+u^\star)}{H_1},\\
b_{23} &= G_1\left(\frac{(1+u^\star)s_2q_5uv}{H_1} + \frac{s_3v^\star(q_6+v)}{H_2} - \frac{a_1q_6w}{H_2}\right),\\
b_{14} &= \frac{q_1}{H_3}(q_2q_3(u-q_4)+q_2u(q_4+u^\star)-a_5p,\\
b_{33} &= a_2 - \frac{a_1v^\star(q_6+v)}{H_2}, \qquad b_{44} = G_3(a_4+a_5u^\star),
\end{aligned}
$$

Now, by using Sylvesters criteria and by choosing $G_1 = \frac{s_1-(u+q_2w(1+u))}{s_2v(1+q_2w)} > 0$ provided $s_1 > (u+q_2w(1+u))$ we get $\frac{dV_1}{dt}$ is negative definite under the following conditions:

$$b_{11} > 0 \tag{19}$$
$$b_{22} > 0 \tag{20}$$
$$b_{33} > 0 \tag{21}$$
$$b_{11}b_{44} > b_{14}^2 \tag{22}$$
$$b_{22}b_{33} > b_{23}^2 \tag{23}$$
$$b_{11}b_{22} > b_{12}^2 \tag{24}$$

It is observed that the fourth inequality, i.e., $b_{11}b_{22} > b_{12}^2$ is satisfied due to the proper choice of G_1, and other inequalities, (14) $\Rightarrow$ 19), (15) $\Rightarrow$ (20), (16) $\Rightarrow$ (21), (18) $\Rightarrow$ (23), (19) $\Rightarrow$ (22). Hence V_1 is a Lyapunov function with respect to $E_{13}^\star$, whose domain contains the region of attraction Ω, proving the theorem.

7 Numerical Simulation

We utilized MATLAB and MATHEMATICA software for numerical simulations. We examine the dynamic behavior of a tri-trophic marine food chain system with "food-limited" affected by environmental toxicants with the help of numerical examples.
We select the following parameter values for E_{10}:

$$
\begin{array}{llllll}
r_0 = 3.05, & r_1 = 1.0, & k = 5.2, & A_1 = 0.05, & B_1 = 1.0, & c = 0.1,\\
C_1 = 0.6, & A_2 = 0.5 & B_2 = 0.9, & D_1 = 0.19, & C_2 = 0.1, & D_2 = 0.05,\\
D_3 = 0.52, & Q = 0.92, & F = 0.31. & & &
\end{array}
$$

It has been determined that, under the above set of parameters, the equilibrium point E_{10}

$$u = 5.2, v = 0, w = 0, p = 0.4080$$

is locally asymptotically stable (see Fig. 1). Now, we select the following parameter values for $\bar{E_{11}}$:

$$r_1 = 1.2, \quad A_1 = 0.3, \quad B_1 = 1.1, \quad c = 4.21 \quad F = 1.51.$$

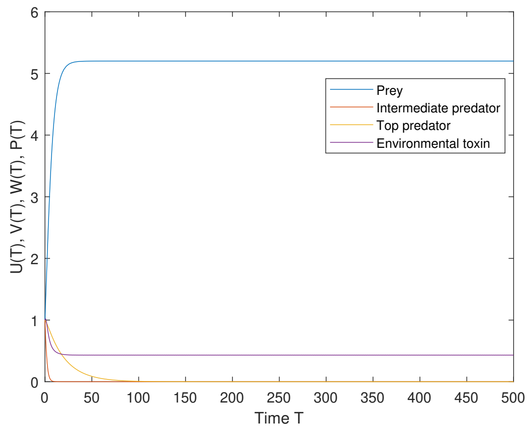

Figure 1: Time series graph around the equilibrium point E_{10}, showing the stability behavior

.

With the above values of parameters and taking the remaining parameters to be the same, it is found that the equilibrium point $\bar{E_{11}}$

$$\bar{u} = 1.7141, \bar{v} = 0, \bar{w} = 0, \bar{p} = 1.25833$$

is locally asymptotically stable (see Fig. 2).
We choose the following parameter values for $\hat{E_{12}}$:

$$\begin{array}{lllllll} r_0 = 3.09, & r_1 = 1.24, & k = 1.2, & A_1 = 0.91, & B_1 = 1.1, & c = 0.81, & C_1 = 2.8596, \\ A_2 = 1.45 & B_2 = 1.71, & D_1 = 0.19, & C_2 = 0.23, & D_2 = 0.45, & D_3 = 8.32, & Q = 1.98, \\ F = 0.991. \end{array}$$

It has been determined that, under the above set of parameters, the equilibrium point $\hat{E_{12}}$

$$\hat{u} = 0.8426, \hat{v} = 0.7863, \hat{w} = 0, \hat{p} = 0.6562$$

is locally asymptotically stable (see Fig. 3).
We select the following value of the parameter for $E^{\star}_{13}$

$$\begin{array}{lllllll} r_0 = 5.092, & r_1 = 1.241, & k = 1.42, & A_1 = 1.612, & B_1 = 0.9812, & c = 1.31, & C_1 = 3.21, \\ A_2 = 2.4521 & B_2 = 0.989, & D_1 = 1.01, & C_2 = 1.5301, & D_2 = 1.345, & D_3 = 2.92, & Q = 0.69, \\ F = 0.21. \end{array}$$

It has been determined that, under the above set of parameters, the equilibrium point $E^{\star}_{13}$

$$u^{\star} = 2.6776, v^{\star} = 0.907941, w^{\star} = 0.4906, p^{\star} = 0.2125$$

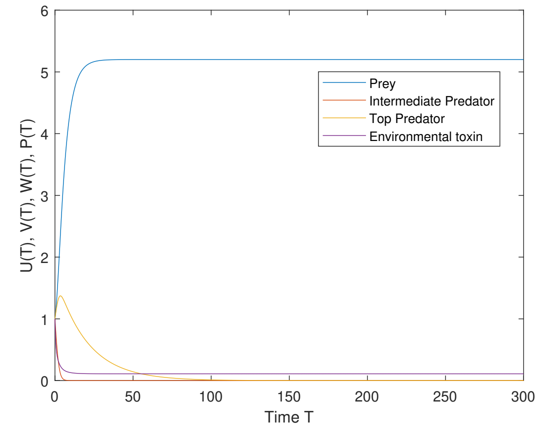

Figure 2: Time series graph around the equilibrium point $\bar{E_{11}}$, showing the stability behavior

.

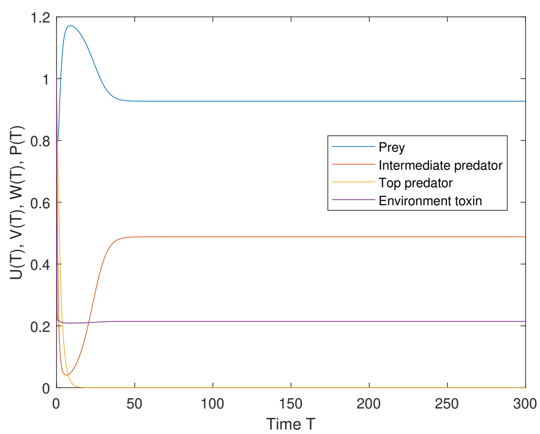

Figure 3: Time series graph around the equilibrium point $\hat{E_{12}}$, showing the stability behavior

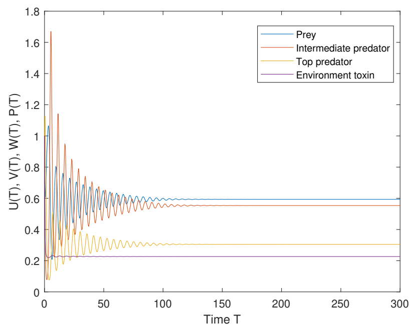

Figure 4: The time graph of the Model illustrates the stability behavior around equilibrium point $E_{13}^{\star}$.

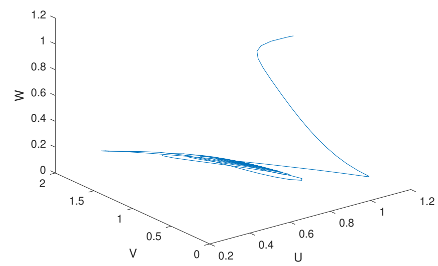

Figure 5: Phase graph of the Model illustrates the stability behavior around equilibrium point $E_{13}^{\star}$.

8 Discussion

In this paper, we have formulated and conducted an analysis of a nonlinear mathematical model to explore the impact of top predator interference on tri-species food chain systems, incorporating "food-limited" growth dynamics under a toxicant environment. The local stability analysis of each equilibrium point in the model has been thoroughly analyzed. The global stability of the nontrivial positive equilibrium point has been analyzed. From the stability of $E_{10}(u,0,0,p)$ and $\bar{E_{11}}(\bar{u},0,0,\bar{p})$ it is concluded that only the prey population will survive and intermediate predator and top predator populations would tend to extinction (see Fig. 1, 2). Stability of $\hat{E_{12}}(\hat{u},\hat{v},0,\hat{p})$ indicates that the prey population and intermediate predator populations will persist, whereas the top predator population is likely to face extinction (see Fig. 3). The interior equilibrium point $E_{13}^{\star}(u^{\star},v^{\star},w^{\star},p^{\star})$ is locally and globally stable, demonstrating the co-existence of all three populations of prey and predator species. It can also be observed from the equilibrium of the intermediate predator population that its level may rise due to a decrease in top predator density, which could be attributed to the effects of toxicants. A comparative analysis has been done in the model with and without toxicants through numerical simulation (see Fig. 6). It is determined that the toxicant-containing system appears to be more stable than the one without any toxicant effects. Additionally, it is noted that as the value of the food parameter increases, the equilibrium level of all three species decreases and for the particular value of ($C = 1.916$), the top predator even may die out (see Fig. 7). We used numerical simulation to show the dynamical behavior of a top predator interference on three-species "food limited" system with the toxicant.

Table 1: Numerical values of the model without toxicant

Equilibrium points	Numerical points
$E_1(u,0,0)$	(5.2,0,0)
$\hat{E_{12}}(\hat{u},\hat{v},0)$	(0.8426,1.4211,0)
$E_{13}^{\star}(u^{\star},v^{\star},w^{\star})$	(2.9677,0.9079,0.4906)

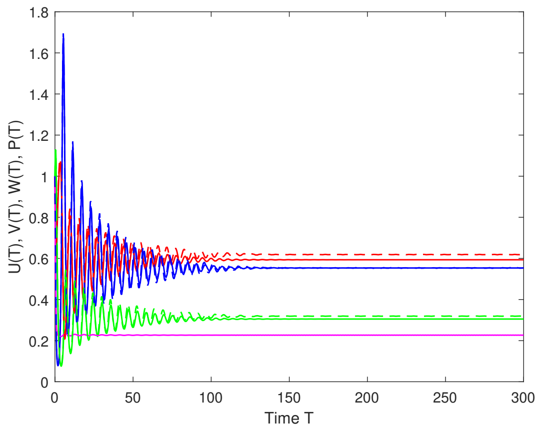

Figure 6: Time graph of the model compared to without toxicant around the equilibrium points $E^{\star}_{13}$ showing the stability behavior.

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Table 2: Numerical values of the model with toxicant

Equilibrium points	Numerical points
$E_{10}(u,0,0,p)$	(5.2,0,0,0.4080)
$\bar{E}_{11}(\bar{u},0,0,\bar{p})$	(1.7141,0,0,1.2583)
$\hat{E}_{12}(\hat{u},\hat{v},0,0,\hat{p})$	(0.8426,0.7863,0,0.6562)
$E^{\star}_{13}(u^{\star},v^{\star},w^{\star},p^{\star})$	(2.6776,0.9079,0.4906,0.2125)

Table 3: Simulation values of the model for different values of parameter C

Equilibrium values of (u,v,w)	parameter C
(1.9441,0.9079,0.4418)	2.46
(1.6271,0.9079,0.4112)	5.86
(1.5580,0.9079,0.4032)	9.163

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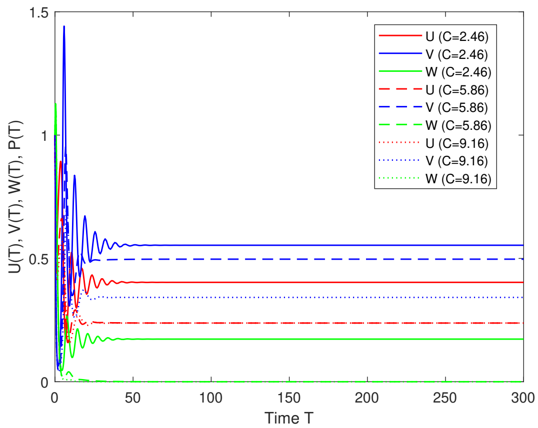

Figure 7: Variation of U, V, W with respect to time T, corresponding to distinct value of C.

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