New Sort of Quotient and Homeomorphisms in Topological Spaces

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Abstract

The aim of this paper is to introduce two new classes of maps called $\delta\ddot{g}$-quotient maps and $\delta\ddot{g}^*$-quotient maps and obtain several characterizations and some of their properties. We further introduce and study new class of generalizations of homeomorphism called $\delta\ddot{g}$-homeomorphism using $\delta\ddot{g}$-closed sets. Also we introduce generalization of homeomorphism called $\delta\ddot{g}c$-homeomorphism. Basic properties of these two mappings are studied and the relation between these types and other existing ones are established.

Keywords and *Phrases* : $\delta\ddot{g}$-closed set, $\delta\ddot{g}$-continuous, $\delta\ddot{g}$-quotient maps, $\delta\ddot{g}$-homeomorphism.
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1 Introduction:

Maki et al [7], introduced the notions of generalized homeomorphism (briefly g-homeomorphism). Devi et al [4] introduced two classes of mappings called generalized semi- homeomorphism (briefly gs- homeomorphism) and semi- generalized homeomorphism (briefly sg-homeomorphism). In this present paper we introduce two new classes of maps called $\delta\ddot{g}$-quotient maps and $\delta\ddot{g}^*$-quotient maps and obtain several characterizations and some of their properties. We further introduce and study new class of generalizations of homeomorphism called $\delta\ddot{g}$-homeomorphism using $\delta\ddot{g}$-closed sets. Also we introduce generalization of homeomorphism called $\delta\ddot{g}c$-homeomorphism. Basic properties of these two mappings are studied and the relation between these types and other existing ones are established.

2 Preliminaries

Throughout this paper (X, τ) and, (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of X, $cl(A)$, $int(A)$ and A^c denote the closure of A, the interior of A and the complement of A respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. The δ-interior [15] of a subset A of X is the union of all regular open set of X contained in A and is denoted by $\text{Int}_\delta(A)$. The subset A is called δ-open [15] if A $=\text{Int}_\delta(A)$, i.e. a set is δ-open if it is the union of regular open sets. the complement of a δ-open is called δ-closed. Alternatively, a set $A \subseteq (X, \tau)$ is called δ-closed [15] if $A = cl_\delta(A)$, where $cl_\delta(A) = \{x : int(cl(U)) \cap A \neq \phi, U \in \tau \text{ and } x \in U\}$.

Definition 2.2. A subset A of (X, τ) is called

(i) semi-generalized closed (briefly sg-closed) set [3] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is a semi-open set in (X, τ).

(ii) generalized semi-closed (briefly gs-closed) set [1] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open set in (X, τ).

(iii) $\delta\hat{g}$-closed set [6] if $\text{cl}_\delta(A) \subseteq$ U whenever A $\subseteq$ U and U is a $\hat{g}$- open set in (X, τ).

(iv) $\delta\ddot{g}$-closed set [9] if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is $\ddot{g}$-set.

The complement of a sg-closed (resp. gs-closed, $\delta\hat{g}$-closed and $\delta\ddot{g}$-closed) set is called sg-open (resp. gs-open, $\delta\ddot{g}$-open).

Definition 2.3. Recall that a function $f : (X, \tau) \to (Y, \tau)$ is called

(i) g-continuous [2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).

(ii) $\delta\hat{g}$-continuous [6] if $f^{-1}(V)$ is $\delta\hat{g}$-closed in (X, τ) for every closed set V of (Y, σ).

(iii) $\delta\hat{g}$-irresolute [6] if $f^{-1}(V)$ is $\delta\hat{g}$-closed in (X, τ) for every $\delta\hat{g}$-closed set V of (Y, σ).

(iv) $\delta\ddot{g}$-continuous[9] if $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ) for every closed set V of (Y,σ).

(v) $\delta\ddot{g}$-irresolute[8] if $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ) for every $\delta\ddot{g}$- closed set V of (Y,σ).

Definition 2.4. A map $f:(X,\tau)\to(Y,\sigma)$ is called

(i) generalized closed (briefly g-closed) (resp. g-open) [12] if the image of every closed (resp. open) set in (X,τ) is g-closed (resp. g-open) in (Y,σ).

(ii) δ-closed [13] if $f(V)$ is δ-closed in (Y,σ) for every δ-closed set V of (X,τ).

(iii) $\delta\hat{g}$-closed [6] if the image of every closed set in (X,τ) is $\delta\hat{g}$-closed in (Y,σ).

(vi) $\delta\ddot{g}$-closed[11] if the image of each closed set in (X,τ) is $\delta\ddot{g}$-closed in (Y,σ).

Definition 2.5. Recall that a map f:(X,τ)$\to$(Y,τ) is called

(i) g-homeomorphism [7] if f is bijection, g-open and g-continuous.

(ii) gs-homeomorphism [4] if f is bijection, gs-open and gs-continuous.

(iii) $\delta\hat{g}$-homeomorphism [6] if f is bijection, $\delta\hat{g}$-open and $\alpha\hat{g}$-continuous.

Definition 2.6. A surjective map $f:(X,\tau)\to(Y,\sigma)$ is said to be

(i) a quotient map [5] , provided a subset V of (Y,σ) is open in (Y, σ) if and only if $f^{-1}(V)$ is open in (X, τ).

(ii) a δ-quotient map [14], provided a subset V of (Y,σ) is δ-open in (Y, σ) if and only if $f^{-1}(V)$ is δ-open in (X, τ).

(iii) a $\delta\hat{g}$-quotient map [8], if f is $\delta\hat{g}$-continuous and f$^{-1}(V)$ is closed in (X,τ) implies V is $\delta\hat{g}$-closed in (Y,σ).

Proposition 2.7. *[9]* Every δ-closed set in X is $\delta\ddot{g}$-closed set.

Proposition 2.8. *[9] Every $\delta\hat{g}$-closed set is $\delta\ddot{g}$-closed.*

3 $\delta\ddot{g}$-Quotient mappings

We introduce the following definition.

Definition 3.1. A surjective map $f:(X,\tau)\to(Y,\sigma)$ is said to be $\delta\ddot{g}$-quotient map if f is $\delta\ddot{g}$-continuous and $f^{-1}(V)$ is closed in (X,τ) implies V is $\delta\ddot{g}$-closed in (Y,σ).

Example 3.2. Let $X=\{a,b,c\}$, $Y=\{p,q,r\}$ with the topologies $\tau=\{\phi,\{b\},\{c\},\{b,c\},\{a,c\},X\}$ and $\sigma=\{\phi,\{p\},\{q\},\{p,q\},\{p,r\},Y\}$. Define a function $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=r$, $f(b)=q$ and $f(c)=p$. Then the function f is $\delta\ddot{g}$-quotient.

Remark 3.3. The concepts of $\delta\ddot{g}$-quotient maps and quotient maps are independent of each other as shown by the following examples.

Example 3.4. Let $X = \{a, b, c\}, Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q\}, \{p, q\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Clearly f is an $\delta\ddot{g}$-quotient map. The set $\{p\}$ is open in $\{Y, \sigma\}$ but $f^{-1}(\{p\}) = \{b\}$ is not open in (X, τ). This implies that f is not continuous and hence f is not an quotient map.

Example 3.5. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{q\}, \{p, q\}, \{q, r\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Clearly f is an quotient map. The set $\{q\}$ is closed in (Y, σ) but $f^{-1}(\{q\}) = \{a\}$ is not $\delta\ddot{g}$-closed in (X, τ). This implies that f is not $\delta\ddot{g}$-continuous and hence f is not an $\delta\ddot{g}$-quotient map.

Theorem 3.6. Every $\delta\hat{g}$-quotient map is $\delta\ddot{g}$-quotient map.

Proof. Suppose $f : (X, \tau) \to (Y, \sigma)$ is an $\delta\hat{g}$-quotient map. Let V be a closed set in $\{Y, \sigma\}$. Since f is $\delta\hat{g}$-continuous, $f^{-1}(V)$ is $\delta\hat{g}$-closed in $\{X, \tau\}$. By Proposition 2.8, $f^{-1}(V)$ is $\delta\ddot{g}$-closed in $\{X, \tau\}$. Therefore f is $\delta\ddot{g}$-continuous. Let $V \subset (Y, \sigma)$ and $f^{-1}(V)$ closed in $\{X, \tau\}$. Then $f(f^{-1}(V)) = V$ is $\delta\hat{g}$-closed set in (Y, σ) and hence V is $\delta\ddot{g}$-closed in (Y, σ). Hence f is $\delta\ddot{g}$-closed map. Thus f is $\delta\ddot{g}$-quotient map. □

Remark 3.7. The converse of the above theorem is not true in general as shown in the following example.

Example 3.8. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$
$\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{q\}, \{p, q\}, \{q, r\}, Y\}$.
Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Clearly f is a $\delta\ddot{g}$-quotient map. The set $\{p, r\}$ is closed in (Y, σ) but $f^{-1}\{p, r\} = \{a, c\}$ is not $\delta\hat{g}$-closed in (X, τ). This implies f is not $\delta\hat{g}$-continuous and hence f is not $\delta\hat{g}$-quotient map.

Remark 3.9. The concepts of $\delta\ddot{g}$-quotient maps and δ-quotient maps are independent of each other as shown in the following examples.

Example 3.10. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, q\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = r$, $f(b) = p$ and $f(c) = q$. Clearly f is a δ-quotient map. However f is not $\delta\ddot{g}$-quotient because $f^{-1}\{r\} = \{a\}$ is not $\delta\ddot{g}$-closed in (X, τ) where $\{r\}$ is closed in (Y, σ).

Example 3.11. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{r\}, \{p, r\}, Y\}$. Define a function

$f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$.Then f is $\delta\ddot{g}$-quotient but not δ-quotient, because $f^{-1}(\{q\}) = \{b\}$ is not δ-closed in (X, τ) where $\{q\}$ is δ-closed in (Y, σ).

Theorem 3.12. Every $\delta\ddot{g}$-quotient map is $\delta\ddot{g}$–closed.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be $\delta\ddot{g}$–quotient map. Let V be a closed set in (X, τ). That is $f^{-1}(f(V))$ is closed in (X, τ). Since f is $\delta\ddot{g}$-quotient, $f(V)$ is $\delta\ddot{g}$-closed in (Y, σ). This shows that f is $\delta\ddot{g}$-closed map. □

Remark 3.13. The converse of the above theorem is not true in general as shown in the following example.

Example 3.14. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{q\}, \{p, r\}, Y\}$. Define function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = r$ and $f(c) = p$. Then f is $\delta\ddot{g}$-closed map. The set $\{q\}$ is closed in (Y, σ) but $f^{-1}(\{q\}) = \{a\}$ not $\delta\ddot{g}$- closed in (X, τ). This implies that f is not $\delta\ddot{g}$- continuous and hence f is not an $\delta\ddot{g}$- quotient map.

Theorem 3.15. Every $\delta\ddot{g}$-quotient map is weakly $\delta\ddot{g}$-closed.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be $\delta\ddot{g}$-quotient map. Let V be δ-closed in (X, τ). That is $f^{-1}(f(V))$ is δ-closed in (X, τ). Every δ-closed is closed and hence $f^{-1}(f(V))$ is closed in (X, τ). Since f is $\delta\ddot{g}$-quotient, $f(V)$ is $\delta\ddot{g}$-closed in (Y, σ). Hence f is weakly $\delta\ddot{g}$-closed map. □

Remark 3.16. The converse of Theorem 3.15 is not true in general. The map f is defined in 3.14 is weakly-$\delta\ddot{g}$-closed but not $\delta\ddot{g}$-quotient.

Proposition 3.17. If $f : (X, \tau) \to (Y, \sigma)$ is surjective, $\delta\ddot{g}$-closed and $\delta\ddot{g}$-continuous. Then f is $\delta\ddot{g}$-quotient map.

Proof. Let $f^{-1}(V)$ be closed in (X, τ). Since f is $\delta\ddot{g}$-closed, $f(f^{-1}(V))$ is $\delta\ddot{g}$-closed set in (Y, σ). Hence V is $\delta\ddot{g}$-closed set, as f is surjective, $f(f^{-1}(V)) = V$. Thus f is an $\delta\ddot{g}$-quotient map. □

Theorem 3.18. Let $f : (X, \tau) \to (Y, \sigma)$ be closed surjective, $\delta\ddot{g}$-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ be an $\delta\ddot{g}$-quotient map. Then $g \circ f$ is an $\delta\ddot{g}$-quotient map.

Proof. Let V be any closed set in (Z, η). Since g is a $\delta\ddot{g}$-quotient map, it is $\delta\ddot{g}$-continuous. So $g^{-1}(V)$ is $\delta\ddot{g}$-closed set in (Y, σ). Since f is $\delta\ddot{g}$-irresolute, $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed set in (X, τ). That is $(g \circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed in (X, τ). This implies $g \circ f$ is $\delta\ddot{g}$-continuous. Also assume that $(g \circ f)^{-1}(V)$ is closed in (X, τ) for $V \subset (Z, \eta)$. That is $f^{-1}(g^{-1}(V))$ is closed in (X, τ). Since f is closed map, $f(f^{-1}(g^{-1}(V)))$ is closed in (Y, σ). That is $g^{-1}(V)$ is closed in (Y, σ) because f is surjective. Since g is $\delta\ddot{g}$-quotient map, V is $\delta\ddot{g}$-closed set in (Z, η). Thus $g \circ f$ is a $\delta\ddot{g}$-quotient map. □

Theorem 3.19. If $f : (X,\tau) \to (Y,\sigma)$ is a $\delta\ddot{g}$–quotient map and $g : (X,\tau) \to (Z,\eta)$ is a continuous map such that it is constant an each set $f^{-1}(\{y\})$ for $y \in Y$. Then g induces an $\delta\ddot{g}$-continuous map $h : (Y,\sigma) \to (Z,\eta)$ such that $h \circ f = g$.

Proof. Since g is constant on $f^{-1}(\{y\})$ for each $y \in Y$, the set $g(f^{-1}(\{y\}))$ is a one point set in Z. If $h(y)$ denote this point, then it is clear that h is well defined and for each $x \in X$, $h(f(x)) = g(x)$. Now we claim that h is $\delta\ddot{g}$-continuous. Let V be closed set in (Z,η). Since g is continuous, $g^{-1}(V)$ is closed in (X,τ). That is $g^{-1}(V) = (h \circ f)^{-1}(V) = f^{-1}(h^{-1}(V))$ is closed in (X,τ). Since f is $\delta\ddot{g}$-quotient map, $h^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Hence h is $\delta\ddot{g}$-continuous. □

We introduce the following definition.

Definition 3.20. A map $f : (X,\tau) \to (Y,\sigma)$ is said to be $\delta\ddot{g}^*$-quotient map if f is surjective, $\delta\ddot{g}$-irresolute and $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ) implies V is closed in (Y,σ).

Example 3.21. Let $X = \{a,b,c\}$, $Y = \{p,q,r\}$ with the topologies $\tau = \{\phi,\{a\},\{b\},\{a,b\},X\}$ and $\sigma = \{\phi,\{p\},\{r\},\{p,r\},Y\}$. Define a map $f : (X,\tau) \to (Y,\tau)$ by $f(a) = p$, $f(b) = r$ and $f(c) = q$. Clearly f is $\delta\ddot{g}^*$-quotient map.

Theorem 3.22. Every $\delta\ddot{g}^*$-quotient map is $\delta\ddot{g}$-irresolute.

Proof. Follows from the definition. □

Remark 3.23. An $\delta\ddot{g}$-irresolute map need not be $\delta\ddot{g}^*$-quotient as the following example shows.

Example 3.24. Let $X = \{a,b,c\}$, $Y = \{p,q,r\}$ with topologies $\tau = \{\phi,\{a\},\{a,c\},X\}$ and $\sigma = \{\phi,\{q,r\},Y\}$. Define a function $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then the function f is not $\delta\ddot{g}^*$-quotient map because $f^{-1}(\{p,q\}) = \{a,b\}$ is $\delta\ddot{g}$-closed in (X,τ) but $\{p,q\}$ is not closed in (Y,σ). However f is $\delta\ddot{g}$-irresolute.

Remark 3.25. The concepts of $\delta\ddot{g}^*$-quotient and $\delta\ddot{g}$-quotient maps are independent of each other as shown by the following examples.

Example 3.26. Let $X = \{a,b,c\}$, $Y = \{p,q,r\}$ with $\tau = \{\phi,\{b\},\{a,b\},\{b,c\},X\}$ and $\sigma = \{\phi,\{r\},\{p,r\},\{q,r\},Y\}$. Define a map $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = p$, $f(b) = r$ and $f(c) = q$. Clearly f is $\delta\ddot{g}^*$-quotient map but not $\delta\ddot{g}$-quotient because $f^{-1}(\{p\}) = \{a\}$ is not $\delta\ddot{g}$-closed in (X,τ) where $\{p\}$ is a closed set in (Y,σ).

Example 3.27. Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{p\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Clearly f is $\delta\ddot{g}$-quotient map but not $\delta\ddot{g}^*$-quotient because $f^{-1}(\{r\}) = \{c\}$ is $\delta\ddot{g}$-closed in (X, τ) but $\{r\}$ is not closed in (Y, σ).

4 $\delta\ddot{g}$-Homeomorphisms

In this section we introduce $\delta\ddot{g}$-homeomorphism and $\delta\ddot{g}c$-homeomorphism. We also discuss some of their properties.

Definition 4.1. A bijection map $f : (X, \tau) \to (Y, \sigma)$ is called $\delta\ddot{g}$-homeomorphism if f is both $\delta\ddot{g}$-continuous and $\delta\ddot{g}$-open.

Theorem 4.2. Every $\delta\ddot{g}$- homeomorphism is gs- homeomorphism.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\delta\ddot{g}$-homeomorphism. Then f is bijective, $\delta\ddot{g}$-continuous and $\delta\ddot{g}$-open map. Let V be an closed set in (Y, σ). Then $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X, τ). Every $\delta\ddot{g}$-closed set is gs-closed and hence, $f^{-1}(V)$ is gs-closed in (X, τ). This implies that f is gs-continuous. Let U be an open set in (X, τ). Then $f(U)$ is $\delta\ddot{g}$-open in (Y, σ). This implies f is gs-open map. Hence f is gs-homeomorphism. □

Remark 4.3. The following example shows that the converse of the above theorem is not be true in general.

Example 4.4. Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, r\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = r$ and $f(c) = q$. Clearly f is gs-homeomorphism but f is not $\delta\ddot{g}$-homeomorphism because $f^{-1}(\{q\}) = \{c\}$ is not a $\delta\ddot{g}$-closed in (X, τ) where $\{q\}$ is closed in (Y, σ).

Theorem 4.5. Every $\delta\ddot{g}$- homeomorphism is g- homeomorphism.

Proof. Follows from the fact that every $\delta\ddot{g}$- continuous map is g- continuous map and every $\delta\ddot{g}$-open map is g-open map. □

Remark 4.6. The converse of the above theorem is not true in general as shown in the following example.

Example 4.7. Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\phi, \{q\}, \{p, q\}, \{q, r\}, Y\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Then obviously f is a g-homeomorphism but f is not $\delta\ddot{g}$-homeomorphism because $f^{-1}(\{p\}) = \{b\}$ is not $\delta\ddot{g}$-closed in (X, τ) where $\{p\}$ is closed in (Y, σ).

Remark 4.8. Homeomorphism and $\delta\ddot{g}$-homeomorphism are independent of each other as shown in the following examples.

Example 4.9. Let $X = \{a, b, c\}$; $Y = \{p, q, r\}$ with $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{r\}, \{p, r\}, \{q, r\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Then f is $\delta\ddot{g}$-open and $\delta\ddot{g}$-continuous. Hence f is a $\delta\ddot{g}$- homeomorphism. However $f^{-1}(\{p, q\}) = \{a, b\}$ is not closed in (X, τ) where $\{p, q\}$ is closed in (Y, σ) and hence f is not continuous. Therefore f is not a homeomorphism.

Example 4.10. Let $X = \{a, b, c\}$; $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{p\}, \{p, q\}, \{p, r\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = q$, $f(b) = p$ and $f(c) = r$. Then f is homeomorphism. The set $\{a, b\}$ is open in (X, τ) but $f(\{a, b\}) = \{p, q\}$ is not $\delta\ddot{g}$-open in (Y, σ). This implies that f is not $\delta\ddot{g}$-open map. Hence f is not a $\delta\ddot{g}$-homeomorphism.

Theorem 4.11. Every $\delta\hat{g}$- homeomorphism is $\delta\ddot{g}$- homeomorphism.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a $\delta\hat{g}$-homeomorphism. Then f is bijective, $\delta\hat{g}$-continuous and $\delta\hat{g}$-open map. Let V be an closed set in (Y, σ). Then $f^{-1}(V)$ is $\delta\hat{g}$-closed in (X, τ). Every $\delta\hat{g}$-closed set is $\delta\ddot{g}$-closed and hence $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X, τ). This implies that f is $\delta\ddot{g}$-continuous. Let U be an open set in (X, τ). Then $f(U)$ is $\delta\hat{g}$-open in (Y, σ). Hence $f(U)$ is $\delta\ddot{g}$-open. This implies f is $\delta\ddot{g}$-open map. Hence f is $\delta\ddot{g}$-homeomorphism. □

Remark 4.12. The following example shows that the converse of the above theorem is not true in general.

Example 4.13. Let $X = \{a, b, c\}$ and $Y = \{p, q, r\}$ with the topologies $\tau = \{\phi, \{a\}, \{b, c\}, , X\}$ and $\sigma = \{\phi, \{q\}, \{p, r\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = p$, $f(b) = q$ and $f(c) = r$. Clearly f is a $\delta\ddot{g}$-homeomorphism but f is not $\delta\hat{g}$-homeomorphism because $f^{-1}(\{p, r\}) = \{a, c\}$ is not $\delta\hat{g}$-closed in (X, τ) where $\{p, r\}$ is closed in (Y, σ).

Proposition 4.14. For any bijective map $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent.

(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\delta\ddot{g}$-continuous map.

(ii) f is an $\delta\ddot{g}$- open map.

(iii) f is an $\delta\ddot{g}$- closed map.

Proof. $(i) \Rightarrow (ii)$. Let U be an open set in (X, τ). Since f^{-1} is $\delta\ddot{g}$-continuous, $(f^{-1})^{-1}(U) = f(U)$ is $\delta\ddot{g}$-open in (Y, σ). Hence f is $\delta\ddot{g}$-open map.
$(ii) \Rightarrow (iii)$. Let F be a closed set in (X, τ). Then F^c is open in (X, τ). Since f is $\delta\ddot{g}$-open map, $f(F^c)$ is $\delta\ddot{g}$-open set in (Y, σ). But $f(F^c) = (f(F))^c$, $(f(F))^c$ is $\delta\ddot{g}$-open in (Y, σ). This implies that $f(F)$ is $\delta\ddot{g}$-closed

in (Y,σ). Hence f is $\delta\ddot{g}$-closed map.
$(iii) \Rightarrow (i)$. Let V be a closed set of (X,τ). Since f is $\delta\ddot{g}$-closed map, $f(V)$ is $\delta\ddot{g}$-closed set in (Y,σ). That is $(f^{-1})^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Hence f^{-1} is $\delta\ddot{g}$-continuous functions. □

Theorem 4.15. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective and $\delta\ddot{g}$-continuous map. Then the following statements are equivalent.

(i) f is an $\delta\ddot{g}$-open map.

(ii) f is an $\delta\ddot{g}$- homeomorphism.

(iii) f is an $\delta\ddot{g}$- closed map.

Proof. $(i) \Rightarrow (ii)$. Let f be a $\delta\ddot{g}$-open map. By hypothesis, f is bijective and $\delta\ddot{g}$-continuous. Hence f is $\delta\ddot{g}$-homeomorphism.
$(ii) \Rightarrow (iii)$. Let f be a $\delta\ddot{g}$-homeomorphism. Then f is $\delta\ddot{g}$-open. By Proposition 4.14, f is $\delta\ddot{g}$-closed map.
$(iii) \Rightarrow (i)$. It is obtained from Proposition 4.14. □

Remark 4.16. The composition of two $\delta\ddot{g}$-homeomorphism need not be $\delta\ddot{g}$-homeomorphism as the following example shows.

Example 4.17. Let $X=\{a,b,c\}=Y=Z$ with the topologies
$\tau=\{\phi,\{a\},\{b,c\},X\}$, $\sigma=\{\phi,\{b\},\{b,c\},Y\}$ and
$\eta=\{\phi,\{a\},\{b\},\{a,b\},\{a,c\},Z\}$. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two identity maps. Then both f and g are $\delta\ddot{g}$-homeomorphism. The set $\{b,c\}$ is open in (X,τ) but $(g\circ f)(\{b,c\})=\{b,c\}$ is not $\delta\ddot{g}$-open in (Z,η). This implies that $g\circ f$ is not $\delta\ddot{g}$-open and hence $g\circ f$ is not $\delta\ddot{g}$-homeomorphism.

Next we introduce the following definition

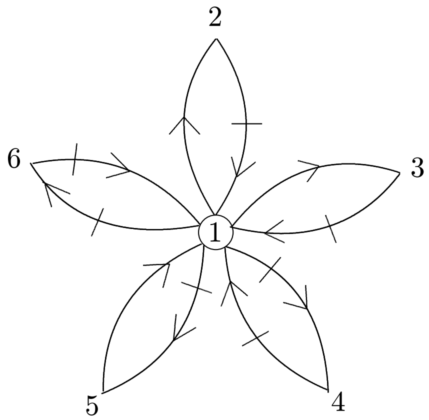
Definition 4.18. A bijection map $f:(X,\tau)\to(Y,\sigma)$ is said to be $\delta\ddot{g}c$-homeomorphism if f is $\delta\ddot{g}$-irresolute and its inverse f^{-1} is $\delta\ddot{g}$-irresolute.

Remark 4.19. $\delta\ddot{g}c$-homeomorphism and $\delta\ddot{g}$- homeomorphisms are independent to each other as shown in the following examples.

Example 4.20. Let $X=\{a,b,c\}=Y$ with the topologies
$\tau=\{\phi,\{b\},\{a,b\},X\}$ and $\sigma=\{\phi,\{b\},Y\}$. Let $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Clearly f is $\delta\ddot{g}$- homeomorphism. The set $\{a,b\}$ is $\delta\ddot{g}$-closed in (Y,σ) but $f^{-1}(\{a,b\})=\{a,b\}$ is not $\delta\ddot{g}$-closed in (X,τ). Therefore f is not $\delta\ddot{g}$-irresolute and hence f is not a $\delta\ddot{g}c$-homeomorphism.

Example 4.21. Let $X=\{a,b,c\}=Y$ with the topologies
$\tau=\{\phi,\{b\},\{a,b\},\{b,c\},X\}$ and $\sigma=\{\phi,\{c\},\{a,c\},\{b,c\},Y\}$. Define a function $f:(X,\tau)\to(Y,\sigma)$ by $f(a)=a$, $f(b)=c$ and $f(c)=b$. Clearly f is $\delta\ddot{g}c$- homeomorphism. The set $\{a,b\}$ is open in (X,τ) but $f(\{a,b\})=\{a,c\}$ is not $\delta\ddot{g}$-open in (Y,σ). This implies that f is not $\delta\ddot{g}$-open map. Then f is not $\delta\ddot{g}$-homeomorphism.

Remark 4.22. From the above discussion we get the following diagram. A → B represents A implies B. A $\nrightarrow$ B represents A does not implies B.



1. $\delta\ddot{g}$-Homeomorphism 2. gs-Homeomorphism 3. g-Homeomorphism 4.Homeomorphism 5. $\delta\hat{g}$-Homeomorphism 6. $\delta\ddot{g}c$-Homeomorphism

Theorem 4.23. The composition of two $\delta\ddot{g}c$-homeomorphism is $\delta\ddot{g}c$-homeomorphism.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two $\delta\ddot{g}c$- homeomorphisms. Let F be a $\delta\ddot{g}$- closed set in (Z, η). Since g is $\delta\ddot{g}$-irresolute map, $g^{-1}(F)$ is $\delta\ddot{g}$- closed in (Y, σ). Since f is $\delta\ddot{g}$- irresolute, $f^{-1}(g^{-1}(F))$ is $\delta\ddot{g}$-closed in (X, τ). That is $(g \circ f)^{-1}(F)$ is $\delta\ddot{g}$-closed in (X, τ). This implies that $g \circ f$ is $\delta\ddot{g}$-irresolute. Let G be a $\delta\ddot{g}$-closed in (X, τ). Since f^{-1} is a $\delta\ddot{g}$- irresolute, $(f^{-1})^{-1}(G) = f(G)$ is $\delta\ddot{g}$-closed in (Y, σ). Since g^{-1} is $\delta\ddot{g}$-irresolute, $(g^{-1})^{-1}(f(G))$ is $\delta\ddot{g}$-closed in (Z, η). That is $g(f(G))$ is $\delta\ddot{g}$- closed in (Z, η). Therefore $(g \circ f)(G)$ is $\delta\ddot{g}$-closed in (Z, η). This implies that $((g \circ f)^{-1})^{-1}(G)$ is $\delta\ddot{g}$-closed in (Z, η). This shows that $(g \circ f)^{-1}$ is $\delta\ddot{g}$-irresolute. Hence $g \circ f$ is $\delta\ddot{g}c$-homeomorphism. □

5 Applications

Definition 5.1. *[9]* A space (X, τ) is called a $T_{\delta\ddot{g}}$-space if every $\delta\ddot{g}$-closed set in it is δ-closed.

Theorem 5.2. Every $\delta\ddot{g}$-quotient map from $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a quotient map.

Proof. Suppose $f : (X, \tau) \to (Y, \sigma)$ is a $\delta\ddot{g}$-quotient map. Let V be a closed set in (Y, σ). Since f is $\delta\ddot{g}$-continuous, $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X, τ). Since (X, τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is closed in (X, τ). Therefore f

is continuous. Let $V \subset (Y,\sigma)$ and $f^{-1}(V)$ be closed in (X,τ) then V is $\delta\ddot{g}$-closed in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, V is closed in (Y,σ). Hence f is quotient map. □

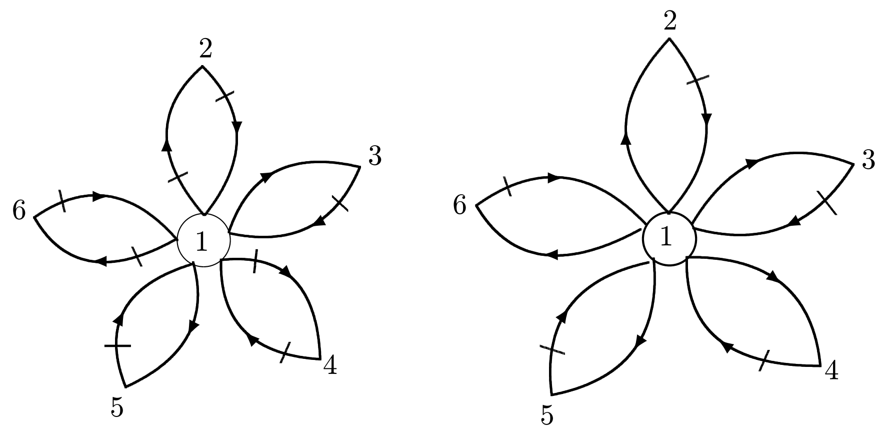
Theorem 5.3. In $T_{\delta\ddot{g}}$-space, every $\delta\ddot{g}$-quotient map is δ-quotient.

Proof. Let V be a δ-closed in (Y,σ). Then V is closed in (Y,σ). Since f is $\delta\ddot{g}$-continuous and (X,τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is δ-closed in (X,τ). Then $f^{-1}(V)$-closed in (X,τ). Since f is $\delta\ddot{g}$-quotient and (X,τ) is $T_{\delta\ddot{g}}$-space, V is δ-closed in (Y,σ). This implies f is δ-quotient map. □

Theorem 5.4. In $T_{\delta\ddot{g}}$-space, every $\delta\ddot{g}$-quotient map is $\delta\ddot{g}^*$-quotient.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be $\delta\ddot{g}$-quotient map. Let V be a $\delta\ddot{g}$-closed set in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space and f is $\delta\ddot{g}$-quotient, $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). This shows that f is $\delta\ddot{g}$-irresolute. Let $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Since (X,τ) is $T_{\delta\ddot{g}}$-space and f is $\delta\ddot{g}$-quotient, V is $\delta\ddot{g}$-closed in (Y,σ). Also since (Y,σ) is $T_{\delta\ddot{g}}$-space, V is closed in (Y,σ). Hence f is $\delta\ddot{g}^*$-quotient map. □

Remark 5.5. From the above discussion, Independency of quotient maps are made dependent quotient maps by applying $T_{\delta\ddot{g}}$-space, seen in the following figures. $A\to$B represents A implies B. $A\nrightarrow B$ represents A does not imply B.



1. $\delta\ddot{g}$-quotient 2. quotient 3. $\delta\hat{g}$-quotient 4. δ-quotient 5. $\delta\ddot{g}$-closed 6. $\delta\ddot{g}^*$-quotient.

Theorem 5.6. Let (Y,σ) be $T_{\delta\ddot{g}}$-space. If $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are $\delta\ddot{g}$-quotient maps. Then their composition $g\circ f:(X,\tau)\to(Z,\eta)$ is a $\delta\ddot{g}$-quotient map.

Proof. Let V be any closed set in $(Z.\eta)$. Since g is $\delta\ddot{g}$-quotient map, it is $\delta\ddot{g}$-continuous. So $g^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, $g^{-1}(V)$ is closed in (Y,σ). Then $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (X,τ), since f is $\delta\ddot{g}$-quotient. That is $(g\circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed in (X, τ). This implies $g\circ f$ is $\delta\ddot{g}$-continuous. Also assume that $(g\circ f)^{-1}(V)$ is closed in (X, τ) for V $\subset (Z,\eta)$. That is $f^{-1}(g^{-1}(V))$ is closed in (X,τ). Since f is $\delta\ddot{g}$-quotient map, $g^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, $g^{-1}(V)$ is closed in (Y,σ). Also since g is $\delta\ddot{g}$-quotient map, V is $\delta\ddot{g}$-closed in (Z,η). Hence $g\circ f$ is $\delta\ddot{g}$-quotient map. □

Theorem 5.7. Let (X,τ) be $T_{\delta\ddot{g}}$-space. If $f:(X,\tau)\to(Y,\sigma)$ is weakly $\delta\ddot{g}$-closed, surjective and $\delta\ddot{g}$-irresolute map and $g:(Y,\sigma)\to(Z,\eta)$ is $\delta\ddot{g}^*$-quotient map. Then $g\circ f:(X,\tau)\to(Z,\eta)$ is $\delta\ddot{g}^*$-quotient map.

Proof. Let V be an $\delta\ddot{g}$-closed set in (Z,η). Since g is $\delta\ddot{g}^*$-quotient, $g^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Since f is $\delta\ddot{g}$-irresolute, $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (X,τ). That is $(g\circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Hence $(g\circ f)$ is $\delta\ddot{g}$-irresolute. Let $(g\circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Then $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (X,τ). Since (X,τ) is $T_{\delta\ddot{g}}$-space and f is weakly $\delta\ddot{g}$-closed map, $f(f^{-1}(g^{-1}(V)))$ is $\delta\ddot{g}$-closed in (Y,σ). That is $g^{-1}(V)$ is $\delta\ddot{g}$-closed in (Y,σ). Since g is $\delta\ddot{g}^*$-quotient, V is closed in (Z,η). Thus $g\circ f$ is $\delta\ddot{g}^*$-quotient map. □

Theorem 5.8. Let $f:(X,\tau)\to(Y,\sigma)$ be $\delta\ddot{g}^*$-quotient and $g:(Y,\sigma)\to(Z,\eta)$ be $\delta\ddot{g}$-closed, surjective and $\delta\ddot{g}$-irresolute where (Z,η) is $T_{\delta\ddot{g}}$-space. Then $g\circ f:(X,\tau)\to(Z,\eta)$ is $\delta\ddot{g}^*$-quotient map.

Proof. Let V be a $\delta\ddot{g}$-closed set in (Z,η). Since g is $\delta\ddot{g}$-irresolute and f is $\delta\ddot{g}^*$-quotient , $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (X,τ). That is $(g\circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Hence $g\circ f$ is $\delta\ddot{g}$-irresolute. Let $(g\circ f)^{-1}(V)$ be $\delta\ddot{g}$-closed in (X,τ). Then $f^{-1}(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (X,τ). Since f is $\delta\ddot{g}^*$-quotient and g is $\delta\ddot{g}$-closed, $g(g^{-1}(V))$ is $\delta\ddot{g}$-closed in (Z,η). That is, V is $\delta\ddot{g}$-closed in (Z,η). Since (Z,η) is $T_{\delta\ddot{g}}$-space, V is closed in (Z,η). Hence $g\circ f$ is $\delta\ddot{g}^*$-quotient. □

Theorem 5.9. Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a homeomorphism.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Then f is bijective, $\delta\ddot{g}$-open map and $\delta\ddot{g}$-continuous. Let U be a open set in (X,τ). Since f is $\delta\ddot{g}$-open and since (Y,σ) is T $_{\delta\ddot{g}}$-space, $f(U)$ is open set in (Y,σ). This implies f is open map. Let V be a closed set in (Y,σ). Since f is $\delta\ddot{g}$-continuous and since (X,τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is closed in (X,τ). Therefore f is continuous. Hence f is a homeomorphism. □

Theorem 5.10. Let (Y,σ) be $T_{\delta\ddot{g}}$-space. If $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ are $\delta\ddot{g}$-homeomorphism then $g\circ f$ is a $\delta\ddot{g}$- homeomorphism.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two $\delta\ddot{g}$- homeomorphism. Let U be an open set in (X,τ). Since f is $\delta\ddot{g}$-open map, $f(U)$ is $\delta\ddot{g}$-open in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, $f(U)$ is open in (Y,σ). Also since g is $\delta\ddot{g}$-open map, $g(f(U))$ is $\delta\ddot{g}$-open in (Z,η). Hence $g\circ f$ is $\delta\ddot{g}$-open map. Let V be a closed set in (Z,η). Since g is $\delta\ddot{g}$-continuous and since (Y,σ) is $T_{\delta\ddot{g}}$-space, $g^{-1}(V)$ is closed in (Y,σ). Since f is $\delta\ddot{g}$-continuous, $f^{-1}(g^{-1}(V))=(g\circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed set in (X,τ). That is $g\circ f$ is $\delta\ddot{g}$-continuous. Hence $g\circ f$ is $\delta\ddot{g}$-homeomorphism. □

Theorem 5.11. Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a $\delta\ddot{g}$-homeomorphism.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Then f is bijective, $\delta\ddot{g}$-open and $\delta\ddot{g}$-continuous maps. Let U be an open set (X,τ). Since f is $\delta\ddot{g}$-open and since (Y,σ) is $T_{\delta\ddot{g}}$-space, $f(U)$ is δ-closed. By Theorem 2.7, every δ-closed set is $\delta\ddot{g}$-closed. Hence $f(U)$ is $\delta\ddot{g}$-closed in (Y,σ). This implies that f is $\delta\ddot{g}$-open. Let V be a closed set in (Y,σ). Since f is $\delta\ddot{g}$-continuous and since (X,τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Therefore f is $\delta\ddot{g}$-continuous. Thus f is $\delta\ddot{g}$-homeomorphism. □

Theorem 5.12. Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a $\delta\ddot{g}c$-homeomorphism.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Let U be $\delta\ddot{g}$-closed in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, U is closed in (Y,σ). Also Since f is $\delta\ddot{g}$-continuous, $f^{-1}(U)$ is $\delta\ddot{g}$-closed in (X,τ). Hence f is $\delta\ddot{g}$-irresolute map. Let V be $\delta\ddot{g}$-open in (X,τ). Since (X,τ) is $T_{\delta\ddot{g}}$-space, V is open in (X,τ). Also since f is $\delta\ddot{g}$-open, $f(V)$ is $\delta\ddot{g}$-open set in (Y,σ). That is $(f^{-1})^{-1}(V)$ is $\delta\ddot{g}$-open in (Y,σ) and hence f^{-1} is $\delta\ddot{g}$-irresolute. Thus f is $\delta\ddot{g}c$- homeomorphism. □

We shall introduce the group structure of the set of all $\delta\ddot{g}c$- homeomorphism from a topological space (X,τ) onto itself by $\delta\ddot{g}c\text{-}h(X,\tau)$.

Theorem 5.13. The set $\delta\ddot{g}c\text{-}h(X,\tau)$ is a group under composition of mappings.

Proof. By Theorem 4.23, $g\circ f\in\delta\ddot{g}c\text{-}h(X,\tau)$ for all $f,g\in\delta\ddot{g}c\text{-}h(X,\tau)$. We know that the composition of mappings is associative. The identity map belonging to $\delta\ddot{g}c\text{-}h(X,\tau)$ acts as the identity element. If $f\in\delta\ddot{g}c\text{-}h(X,\tau)$ then $f^{-1}\in\delta\ddot{g}c\text{-}h(X,\tau)$ such that $f\circ f^{-1}=f^{-1}\circ f=I$ and so inverse exists for each element of $\delta\ddot{g}c\text{-}h(X,\tau)$. Hence $\delta\ddot{g}c\text{-}h(X,\tau)$ is a group under the composition of mappings. □

Theorem 5.14. Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta\ddot{g}c$-homeomorphism. Then f induces an isomorphism from the group $\delta\ddot{g}c\text{-}h(X,\tau)$ onto the group $\delta\ddot{g}c\text{-}h(Y,\sigma)$.

Proof. We define a map $f_* : \delta\ddot{g}c\text{-}h(X,\tau) \to \delta\ddot{g}c\text{-}h(Y,\sigma)$ by $f_*(k) = f \circ k \circ f^{-1}$ for every $k \in \delta\ddot{g}c\text{-}h(X,\tau)$. Then f_* is a bijection and also for all $k_1, k_2 \in \delta\ddot{g}c\text{-}h(X,\tau)$, $f_*(k_1 \circ k_2) = f \circ (k_1 \circ k_2) \circ f^{-1} = (f \circ k_1 \circ f^{-1}) \circ (f \circ k_2 \circ f^{-1}) = f_*(k_1) \circ f_*(k_2)$. Hence f_* is a homeomorphism and so it is an isomorphism induced by f. □

Theorem 5.15. *Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Then f is bijective, $\delta\ddot{g}$-open map and $\delta\ddot{g}$-continuous. Let U be a open set in (X,τ). Since f is $\delta\ddot{g}$-open and since (Y,σ) is T$_{\delta\ddot{g}}$-space, $f(U)$ is open set in (Y,σ). This implies f is open map. Let V be a closed set in (Y,σ). Since f is $\delta\ddot{g}$-continuous and since (X,τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is closed in (X,τ). Therefore f is continuous. Hence f is a homeomorphism. □

Theorem 5.16. *Let (Y,σ) be $T_{\delta\ddot{g}}$-space. If $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ are $\delta\ddot{g}$-homeomorphism then $g \circ f$ is a $\delta\ddot{g}$- homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ be two $\delta\ddot{g}$- homeomorphism. Let U be an open set in (X,τ). Since f is $\delta\ddot{g}$-open map, $f(U)$ is $\delta\ddot{g}$-open in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, $f(U)$ is open in (Y,σ). Also since g is $\delta\ddot{g}$-open map, $g(f(U))$ is $\delta\ddot{g}$-open in (Z,η). Hence $g \circ f$ is $\delta\ddot{g}$-open map. Let V be a closed set in (Z,η). Since g is $\delta\ddot{g}$-continuous and since (Y,σ) is $T_{\delta\ddot{g}}$-space, $g^{-1}(V)$ is closed in (Y,σ). Since f is $\delta\ddot{g}$-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\delta\ddot{g}$-closed set in (X,τ). That is $g \circ f$ is $\delta\ddot{g}$-continuous. Hence $g \circ f$ is $\delta\ddot{g}$-homeomorphism. □

Theorem 5.17. *Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a $\delta\ddot{g}$-homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Then f is bijective, $\delta\ddot{g}$-open and $\delta\ddot{g}$-continuous maps. Let U be an open set (X,τ). Since f is $\delta\ddot{g}$-open and since (Y,σ) is $T_{\delta\ddot{g}}$-space, $f(U)$ is δ-closed. By Proposition 2.7, every δ-closed set is $\delta\ddot{g}$-closed. Hence $f(U)$ is $\delta\ddot{g}$-closed in (Y,σ). This implies that f is $\delta\ddot{g}$-open. Let V be a closed set in (Y,σ). Since f is $\delta\ddot{g}$-continuous and since (X,τ) is $T_{\delta\ddot{g}}$-space, $f^{-1}(V)$ is $\delta\ddot{g}$-closed in (X,τ). Therefore f is $\delta\ddot{g}$-continuous. Thus f is $\delta\ddot{g}$-homeomorphism. □

Theorem 5.18. *Every $\delta\ddot{g}$-homeomorphism from a $T_{\delta\ddot{g}}$-space in to another $T_{\delta\ddot{g}}$-space is a $\delta\ddot{g}c$-homeomorphism.*

Proof. Let $f : (X,\tau) \to (Y,\sigma)$ be a $\delta\ddot{g}$-homeomorphism. Let U be $\delta\ddot{g}$-closed in (Y,σ). Since (Y,σ) is $T_{\delta\ddot{g}}$-space, U is closed in (Y,σ). Also Since f is $\delta\ddot{g}$-continuous, $f^{-1}(U)$ is $\delta\ddot{g}$-closed in (X,τ). Hence f is $\delta\ddot{g}$-irresolute map. Let V be $\delta\ddot{g}$-open in (X,τ). Since (X,τ) is $T_{\delta\ddot{g}}$-space, V is open in (X,τ). Also since f is $\delta\ddot{g}$-open, $f(V)$ is $\delta\ddot{g}$-open set in (Y,σ). That is

$(f^{-1})^{-1}(V)$ is $\delta\ddot{g}$-open in (Y,σ) and hence f^{-1} is $\delta\ddot{g}$-irresolute. Thus f is $\delta\ddot{g}c$- homeomorphism. ☐

We shall introduce the group structure of the set of all $\delta\ddot{g}c$- homeomorphism from a topological space (X,τ) onto itself by $\delta\ddot{g}c\text{-}h(X,\tau)$.

Theorem 5.19. *The set $\delta\ddot{g}c\text{-}h(X,\tau)$ is a group under composition of mappings.*

Proof. By Theorem 4.23, $g\circ f\in\delta\ddot{g}c\text{-}h(X,\tau)$ for all $f,g\in\delta\ddot{g}c\text{-}h(X,\tau)$. We know that the composition of mappings is associative. The identity map belonging to $\delta\ddot{g}c\text{-}h(X,\tau)$ acts as the identity element. If $f\in\delta\ddot{g}c\text{-}h(X,\tau)$ then $f^{-1}\in\delta\ddot{g}c\text{-}h(X,\tau)$ such that $f\circ f^{-1}=f^{-1}\circ f=I$ and so inverse exists for each element of $\delta\ddot{g}c\text{-}h(X,\tau)$. Hence $\delta\ddot{g}c\text{-}h(X,\tau)$ is a group under the composition of mappings. ☐

Theorem 5.20. *Let $f:(X,\tau)\to(Y,\sigma)$ be a $\delta\ddot{g}c$-homeomorphism. Then f induces an isomorphism from the group $\delta\ddot{g}c\text{-}h(X,\tau)$ onto the group $\delta\ddot{g}c\text{-}h(Y,\sigma)$.*

Proof. We define a map $f_*:\delta\ddot{g}c\text{-}h(X,\tau)\to\delta\ddot{g}c\text{-}h(Y,\sigma)$ by $f_*(k)=f\circ k\circ f^{-1}$ for every $k\in\delta\ddot{g}c\text{-}h(X,\tau)$. Then f_* is a bijection and also for all $k_1,k_2\in\delta\ddot{g}c\text{-}h(X,\tau)$, $f_*(k_1\circ k_2)=f\circ(k_1\circ k_2)\circ f^{-1}=(f\circ k_1\circ f^{-1})\circ(f\circ k_2\circ f^{-1})=f_*(k_1)\circ f_*(k_2)$. Hence f_* is a homeomorphism and so it is an isomorphism induced by f. ☐

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